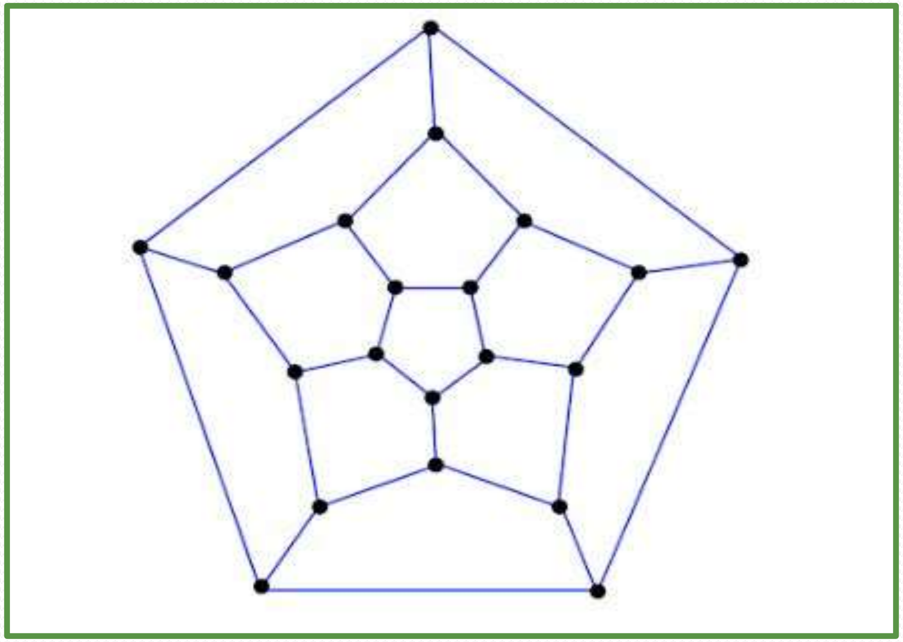
Hamilton Circuits/Graphs



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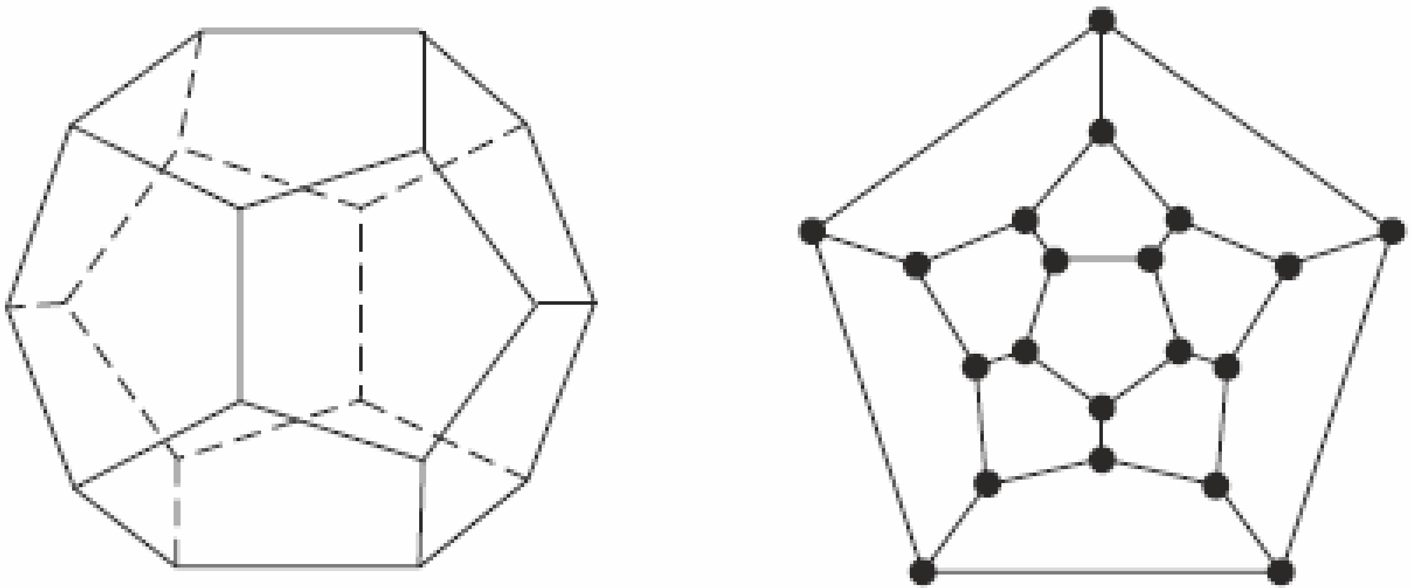
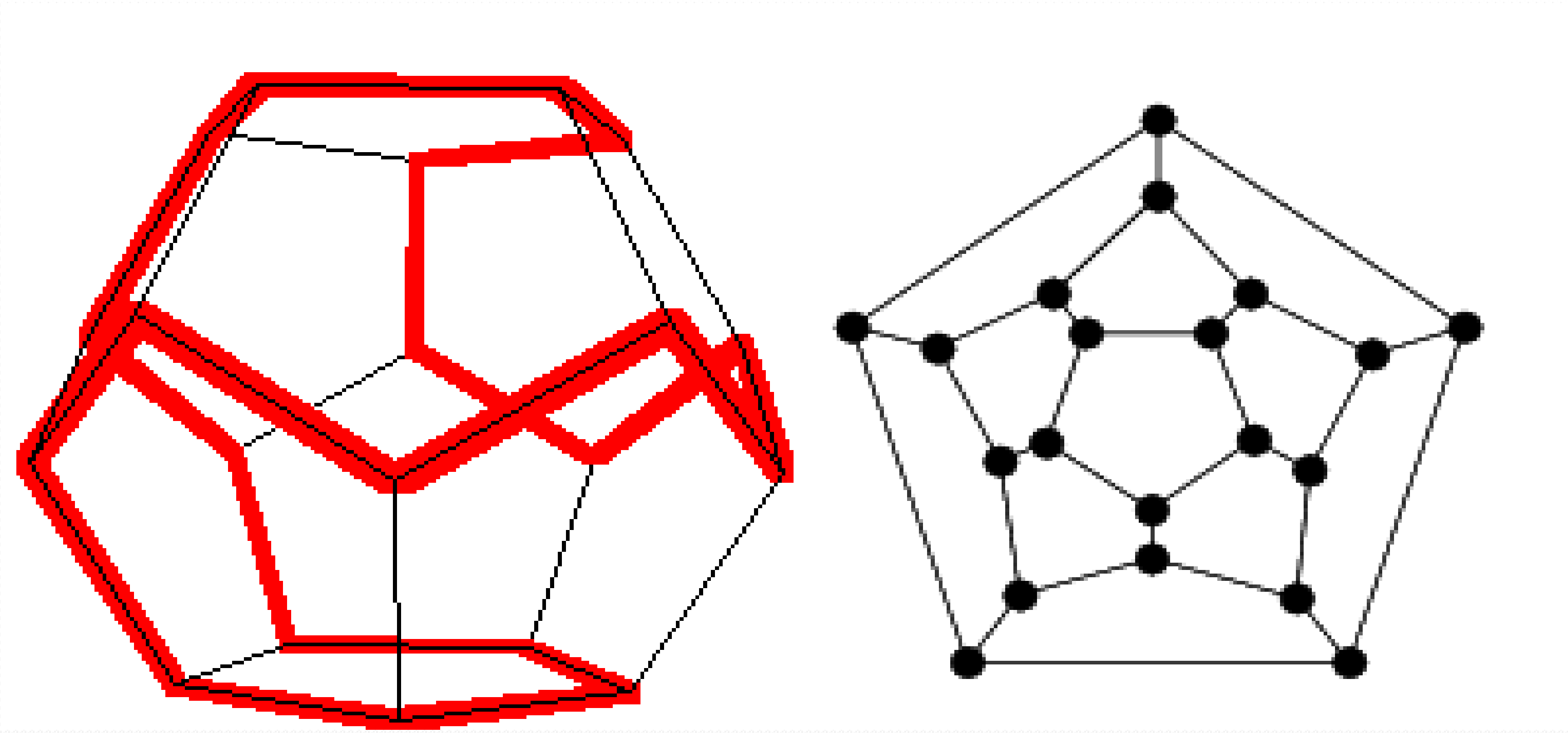
Hamilton Circuits



Dodecahedron puzzle and it equivalent graph

Is there a circuit in this graph that passes through each vertex exactly once?

Hamilton Circuits

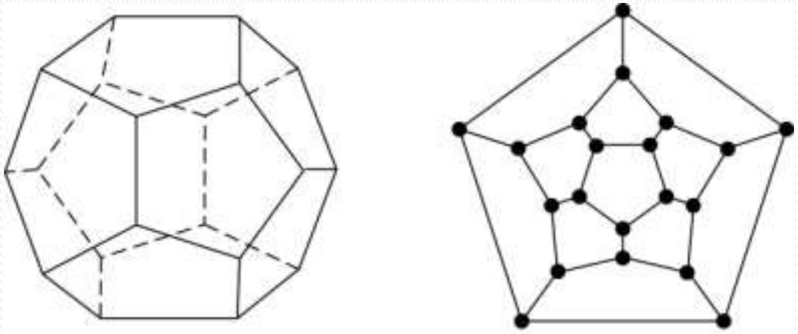


Yes; this is a circuit that passes through each vertex exactly once.

Hamilton Paths and Circuits

William Rowan Hamilton (1805- 1865)

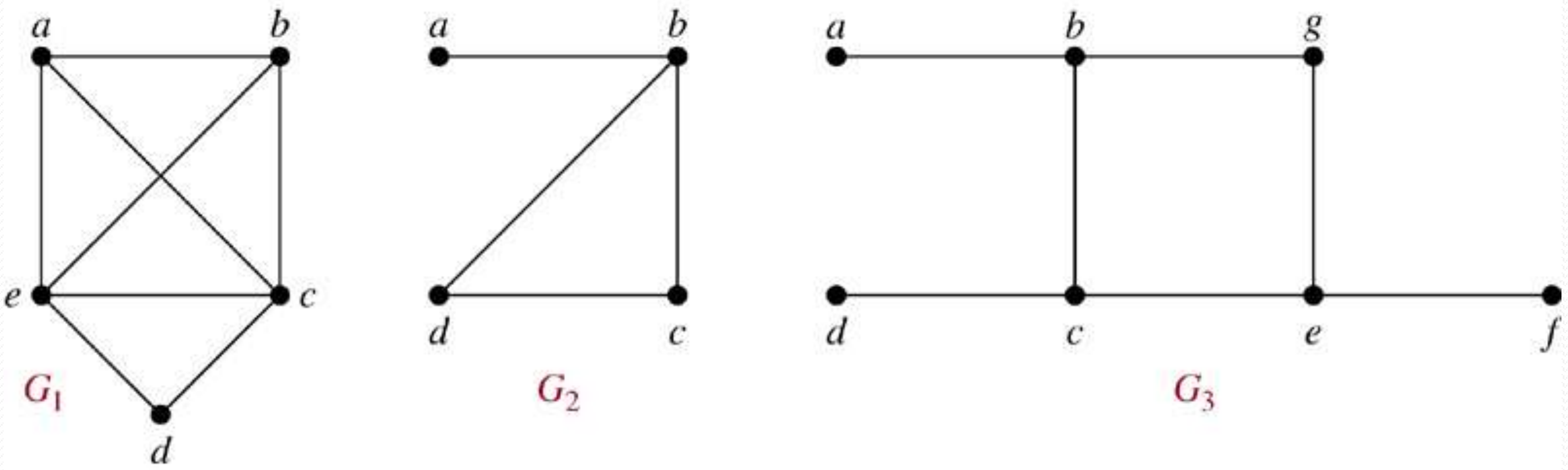
- Euler paths and circuits contained every edge only once. Now we look at paths and circuits that contain every vertex exactly once.
- William Hamilton invented the *Icosian puzzle* in 1857. It consisted of a wooden dodecahedron (with 12 regular pentagons as faces), illustrated in (a), with a peg at each vertex, labeled with the names of different cities. String was used to used to plot a circuit visiting 20 cities exactly once
- The graph form of the puzzle is given in (b).



(a) (b)

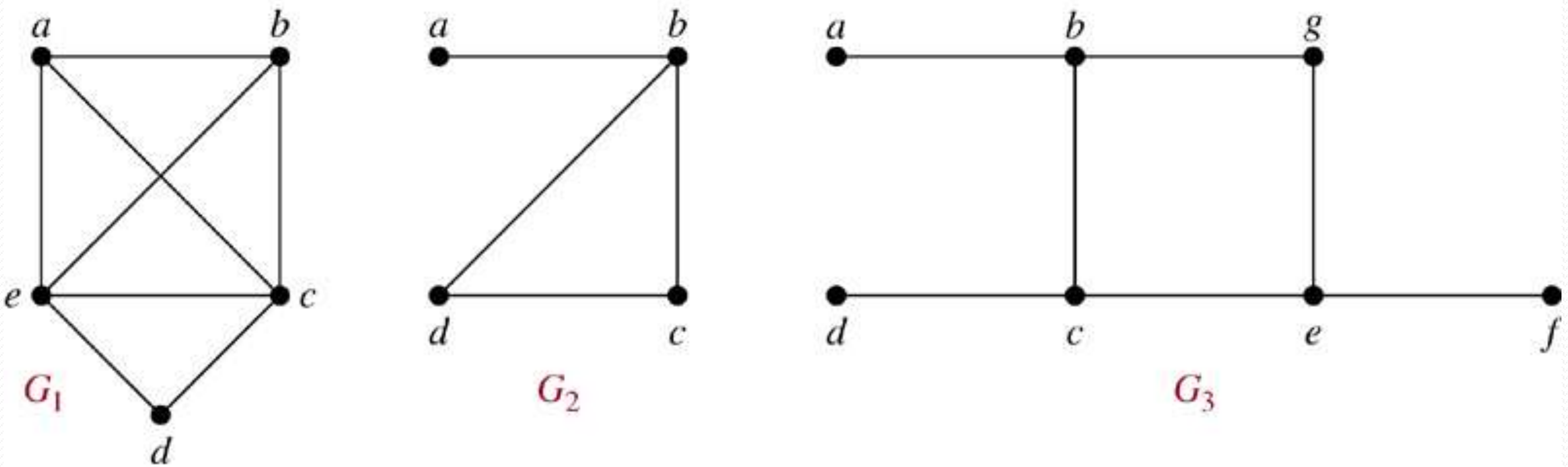
Definition: A simple path in a graph G that passes through every vertex exactly once is called a *Hamilton path*, and a simple circuit in a graph G that passes through every vertex exactly once is called a *Hamilton circuit*.

Finding Hamilton Circuits



Which of these three figures has a Hamilton circuit?
Or, if no Hamilton circuit, a Hamilton path?

Finding Hamilton Circuits



- G_1 has a Hamilton circuit: a, b, c, d, e, a
- G_2 does not have a Hamilton circuit, but does have a Hamilton path: a, b, c, d
- G_3 has neither.

Euler versus Hamilton

Property	Euler	Hamilton
Repeated visits to a given node allowed?	Yes	No
Repeated traversals of a given edge allowed?	No	No
Omitted nodes allowed?	No	No
Omitted edges allowed?	No	Yes

Sufficient Conditions for Hamiltonian Circuits

Gabriel Andrew Dirac
(1925-1984)

- Unlike for an Euler circuit, no simple necessary and sufficient conditions are known for the existence of a Hamiltonian circuit.
- However, there are some useful sufficient conditions. We describe two of these now.
- NOTE: These are not necessary conditions for a graph to be a hamiltonian

Dirac's Theorem:
If G is a simple graph with $n \geq 3$ vertices such that the degree of every vertex in G is $\geq n/2$, then G has a Hamilton circuit.

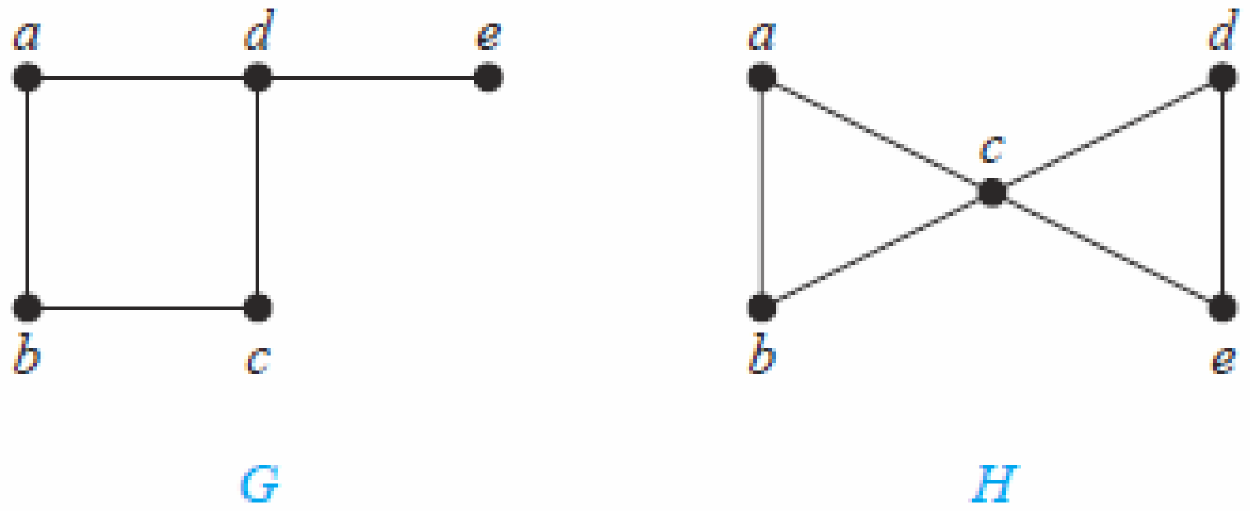
Ore's Theorem: I
f G is a simple graph with $n \geq 3$ vertices such that deg(u) + deg(… r every pair of nonadjacent vertices, then G has a Hamilton circuit.

Øysten Ore
(1899-1968)

Properties to look for ...

- No vertex of degree 1
- No cut edges
- If a node has degree 2, then both edges incident to it must be in any Hamilton circuit.
- No smaller circuits contained in any Hamilton circuit (the start/endpoint of any smaller circuit would have to be visited twice).

Show that neither graph displayed below has a Hamilton circuit.



There is no Hamilton circuit in G because G has a vertex of degree one: e.

Now consider *H. Because the degrees of the vertices a, b, d, and e are all two, every edge* incident with these vertices must be part of any Hamilton circuit. No Hamilton circuit can exist in *H,* for any Hamilton circuit would have to contain four edges incident with *c,* which is impossible.

Time Complexity

The best algorithms known for finding a Hamilton circuit in a graph or determining that no such circuit exists have exponential worst-case time complexity (in the number of vertices of the graph).

Finding an algorithm that solves this problem with polynomial worst-case time complexity would be a major accomplishment because it has been shown that **this problem is NP-complete**. Consequently, the existence of such an algorithm would imply that many other seemingly intractable problems could be solved using algorithms with polynomial worst-case time complexity.

Applications of Hamilton Paths and Circuits

- Applications that ask for a path or a circuit that visits each intersection of a city, each place pipelines intersect in a utility grid, or each node in a communications network exactly once, can be solved by finding a Hamilton path in the appropriate graph.

- The famous *traveling salesperson problem* (*TSP*) asks for the shortest route a traveling salesperson should take to visit a set of cities. This problem reduces to finding a Hamilton circuit such that the total sum of the weights of its edges is as small as possible.