Particle Swarm optimization

Contents

Introduction to Optimization

- The optimization can be defined as a mechanism through which the maximum or minimum value of a given function or process can be found.

- The function that we try to minimize or maximize is called as objective function.
- Variable and parameters.
- Statement of optimization problem

Minimize f(x)

subject to g(x)<=0

h(x)=0.

- Two main phases **Exploration and Exploitation**

Introduction to Optimization

Application to optimization: Particle Swarm Optimization

Proposed by James Kennedy & Russell Eberhart (1995)

Combines self-experiences with social experiences

Particle Swarm Optimization(PSO)

- Inspired from the nature social behavior and dynamic movements with communications of insects, birds and fish.

Particle Swarm Optimization(PSO)

- ✓ Uses a number of agents (**particles**) that constitute a swarm moving around in the search space looking for the best solution

- ✓ Each particle in search space adjusts its “flying” according to its own flying experience as well as the flying experience of other particles.

- ✓ Each particle has three parameters position, velocity, and previous best position, particle with best fitness value is called as global best position.

Contd..

- ✓ Collection of flying particles (swarm) - Changing solutions

Search area - Possible solutions

- ✓ Movement towards a promising area to get the global optimum.
- ✓ Each particle adjusts its travelling speed dynamically corresponding to the flying experiences of itself and its colleagues.
- ✓ Each particle keeps track:

  its best solution, personal best, *pbest.*

  the best value of any particle, global best, *gbest.*

- ✓ Each particle modifies its position according to:
  - its current position
  - its current velocity
  - the distance between its current position and pbest.
  - the distance between its current position and gbest.

Algorithm - Parameters

***f* : Objective function**

Xi: Position of the particle or agent.

Vi: Velocity of the particle or agent.

A: Population of agents.

W: Inertia weight.

C1: cognitive constant.

R1, R2: random numbers.

C2: social constant.

Algorithm - Steps

1. Create a 'population' of agents (particles) uniformly distributed over X

2. Evaluate each particle's position according to the objective function(say

$$Y=F(x) = -x^2+5x+20$$

1. If a particle's current position is better than its previous best position, update it.

2. Determine the best particle (according to the particle's previous best positions).

Contd..

5. Update particles' velocities:

$$\mathbf{v}_i^{t+1} = \underbrace{\mathbf{v}_i^t}_{inertia} + \underbrace{c_1 \mathbf{U}_1^t (\mathbf{pb}_i^t - \mathbf{p}_i^t)}_{personal\ influence} + \underbrace{c_2 \mathbf{U}_2^t (\mathbf{gb}^t - \mathbf{p}_i^t)}_{social\ influence}$$

6. Move particles to their new positions:

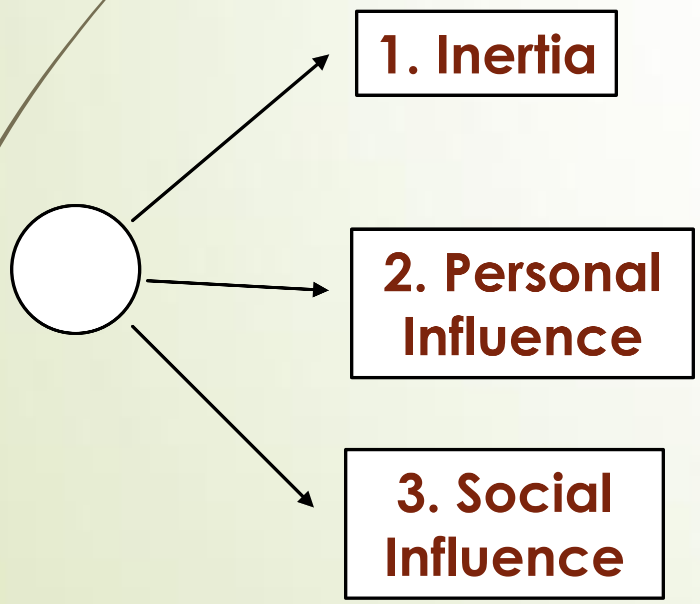
$$\mathbf{p}_i^{t+1} = \mathbf{p}_i^t + \mathbf{v}_i^{t+1}$$

7. Go to step 2 until stopping criteria are satisfied.

Contd…

Particle’s velocity

$$\mathbf{v}_i^{t+1} = \underbrace{\mathbf{v}_i^{t}}_{inertia} + \underbrace{c_1 \mathbf{U}_1^{t}(\mathbf{pb}_i^{t} - \mathbf{p}_i^{t})}_{personal\ influence} + \underbrace{c_2 \mathbf{U}_2^{t}(\mathbf{gb}^{t} - \mathbf{p}_i^{t})}_{social\ influence}$$



- Makes the particle move in the same direction and with the same velocity

- Improves the individual
- Makes the particle return to a previous position, better than the current
- Conservative

- Makes the particle follow the best neighbors direction

Acceleration coefficients

- When , **c1=c2=0** then all particles continue flying at their current speed until they hit the search space's boundary. Therefore, the velocity update equation is calculated as:

$$v_{ij}^{t+1} = v_{ij}^{t}$$

- When **c1>0 and c2=0** , all particles are independent. The velocity update equation will be:

$$v_{ij}^{t+1} = v_{ij}^{t} + c1r1_{j}^{t}\left[P^{t}_{best,i} - x_{ij}^{t}\right]$$

- When c1>0 and c2=0 , all particles are attracted to a single point in the entire swarm and the update velocity will become

$$v_{ij}^{t+1} = v_{ij}^{t} + c2r2_{j}^{t}\left[g_{best} - x_{ij}^{t}\right]$$

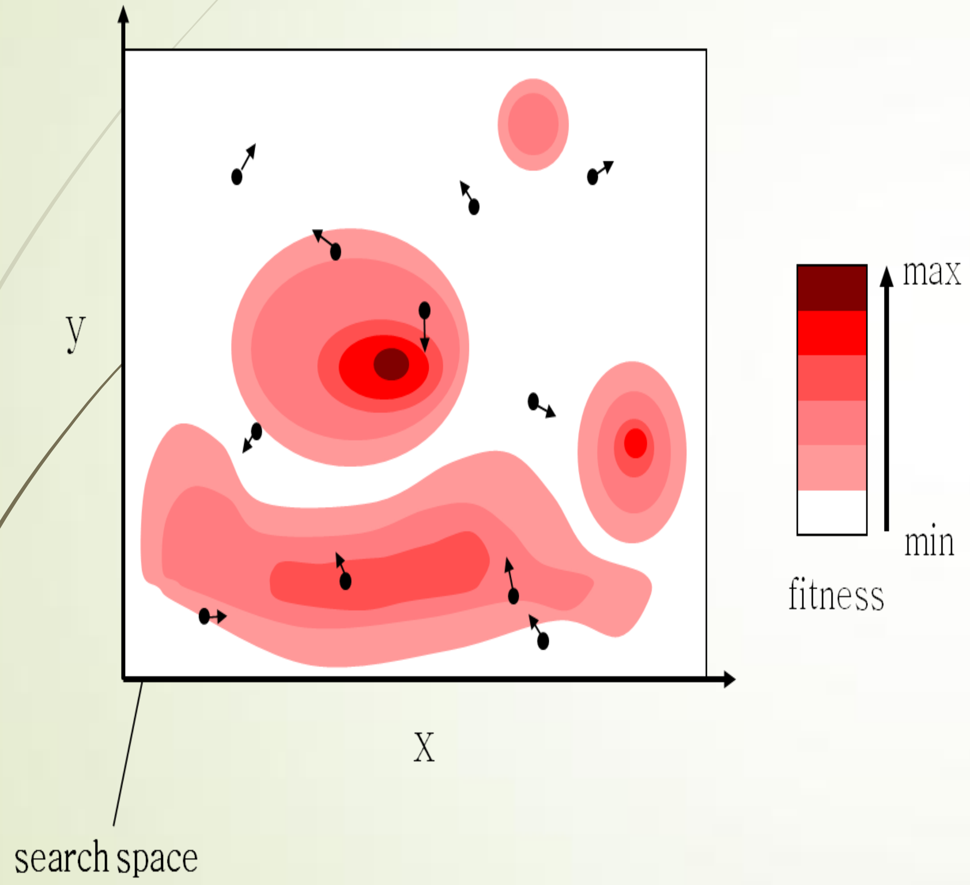
- When c1=c2, all particles are attracted towards the average of pbest and gbest.

Contd…

- **Intensification**: explores the previous solutions, finds the best solution of a given region
- **Diversification**: searches new solutions, finds the regions with potentially the best solutions
- **In PSO:**

$$\underbrace{\mathbf{v}_i^{t+1} = \mathbf{v}_i^t}_{\text{Diversification}} + \underbrace{c_1 \mathbf{U}_1^t(\mathbf{pb}_i^t - \mathbf{p}_i^t) + c_2 \mathbf{U}_2^t(\mathbf{gb}^t - \mathbf{p}_i^t)}_{\text{Intensification}}$$

Example:



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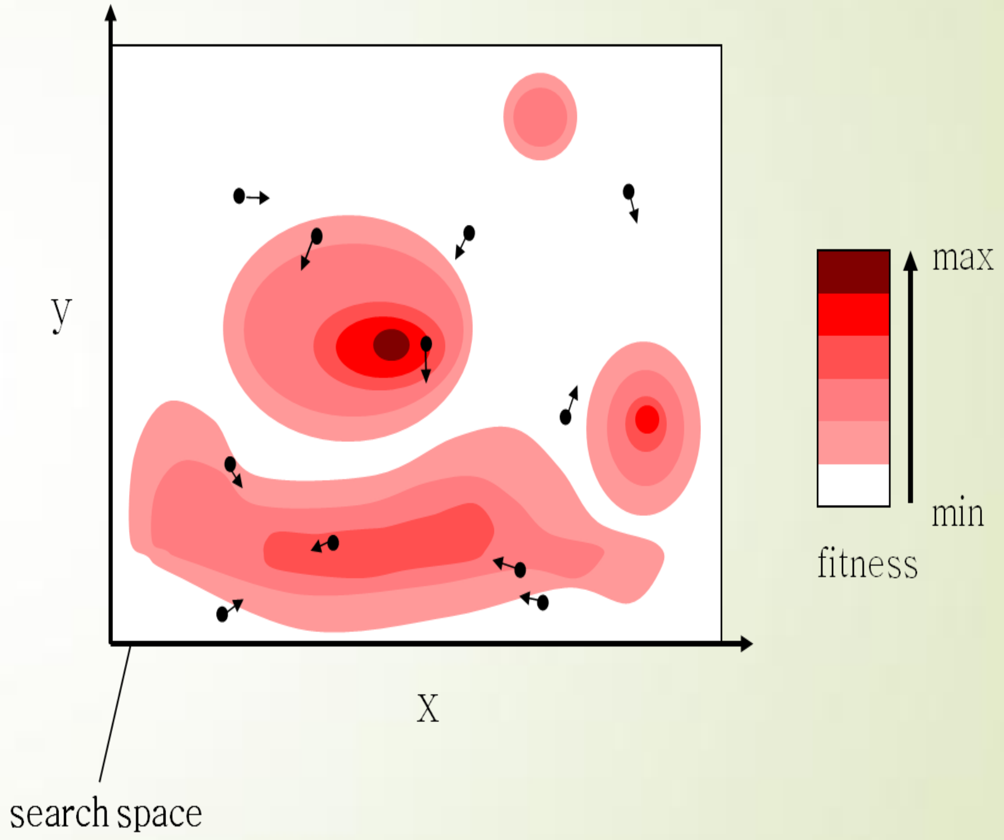
fitness
X
search space



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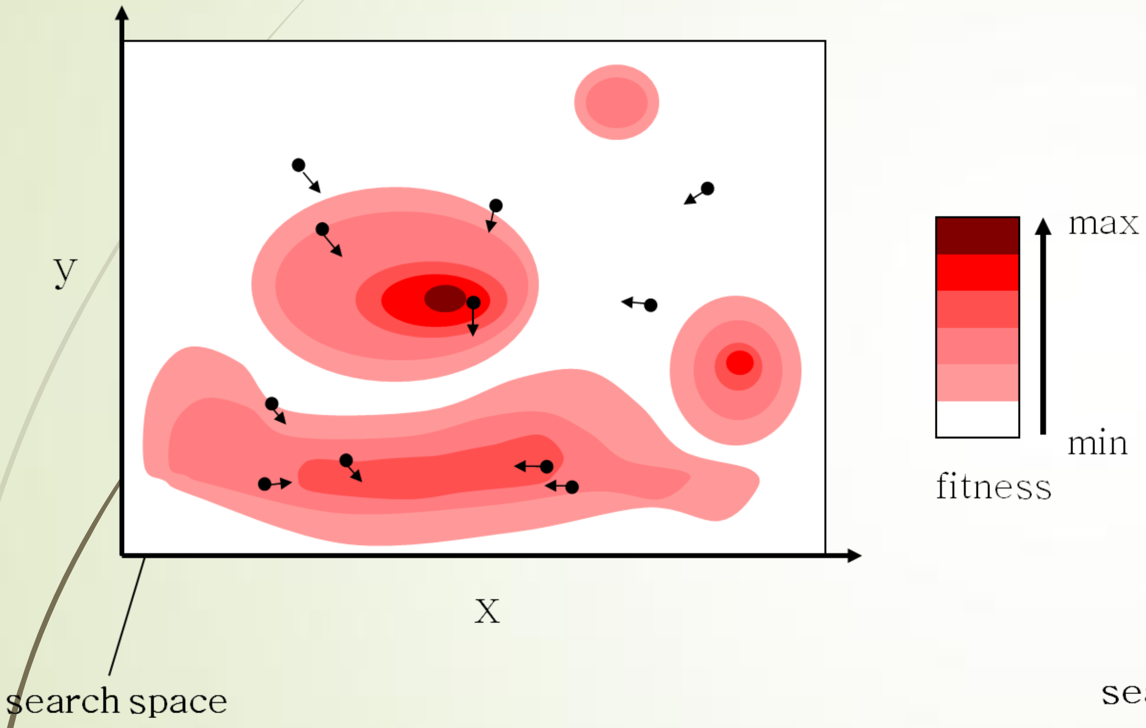
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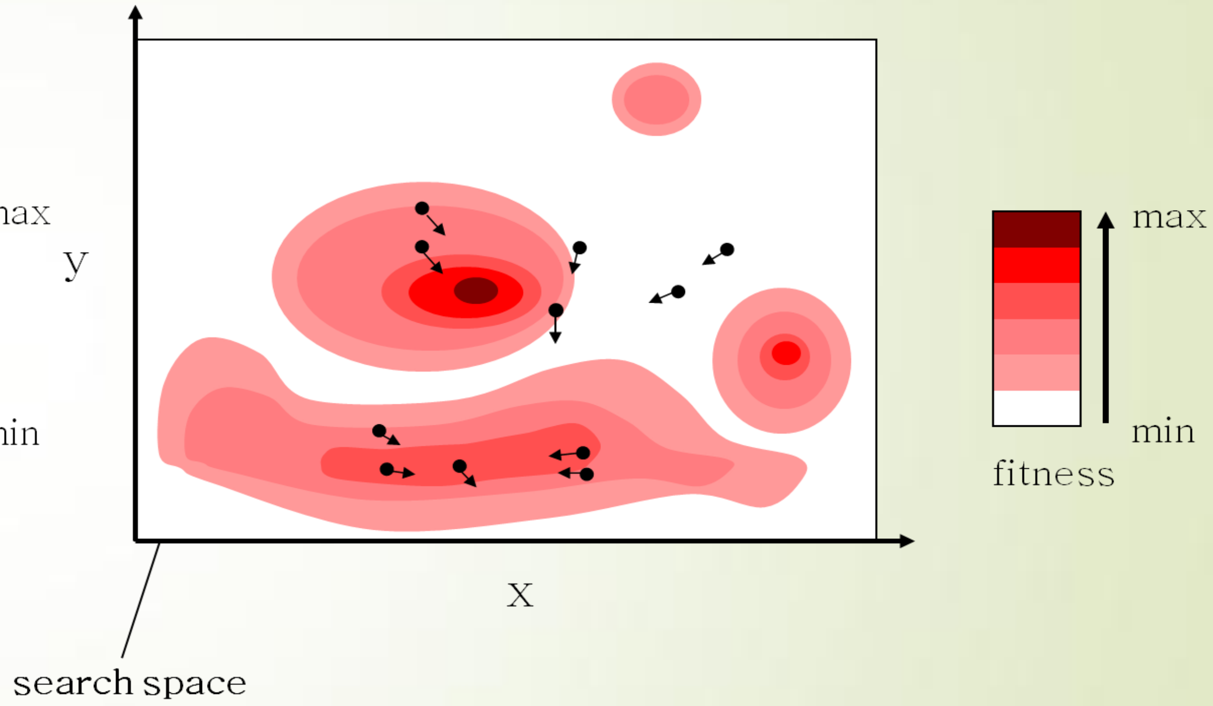
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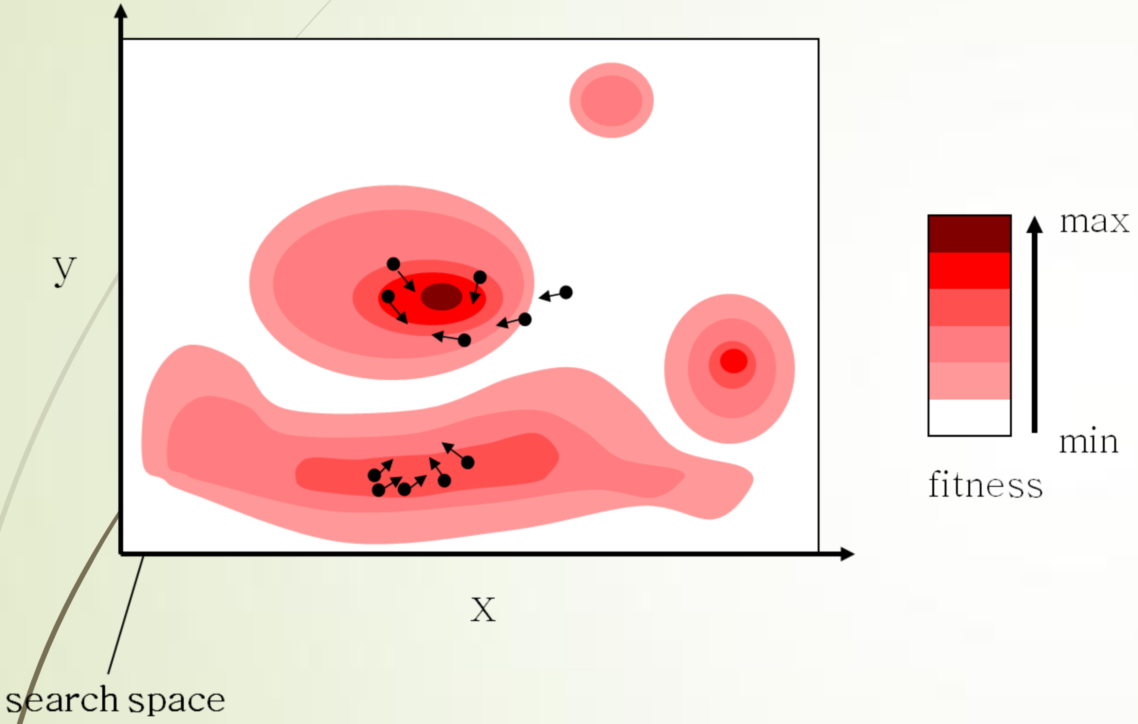
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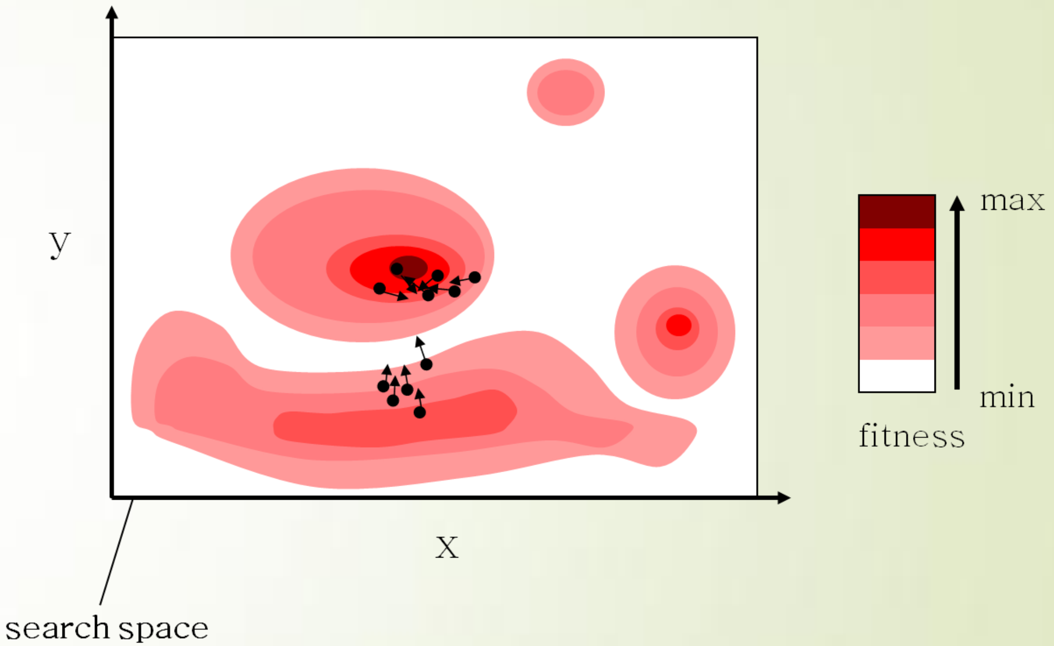
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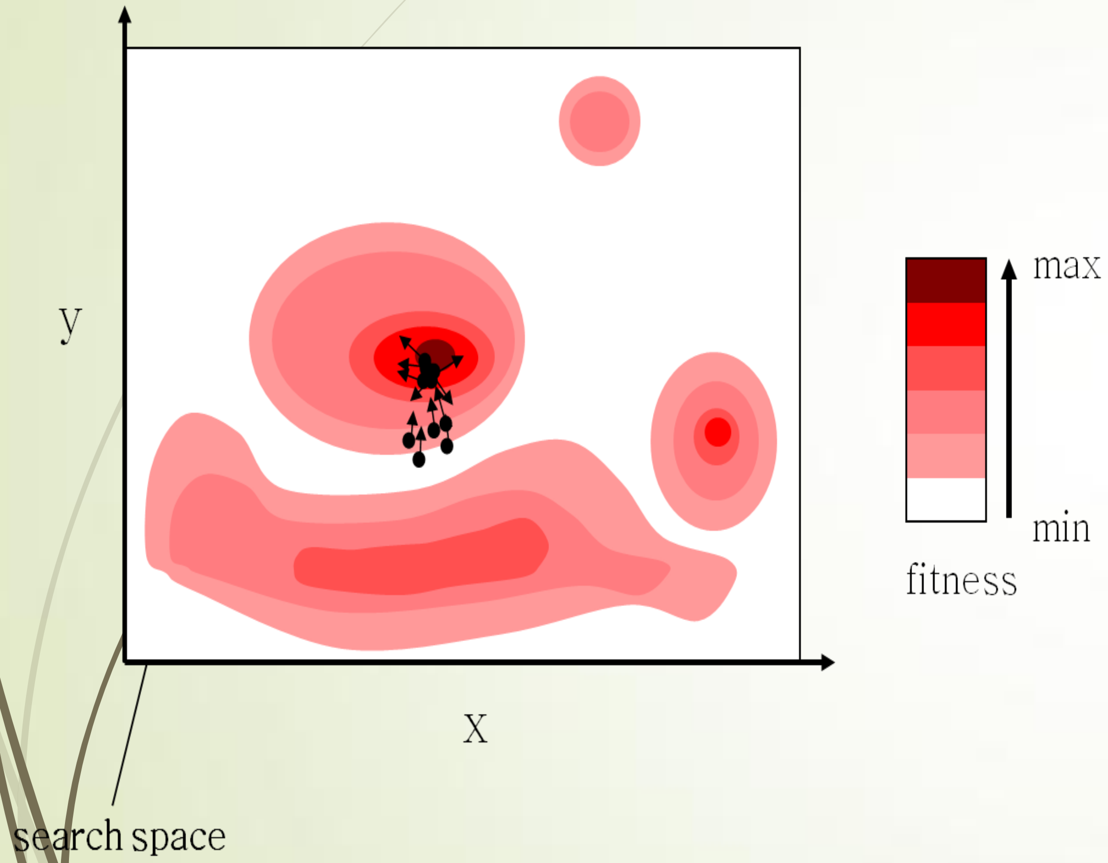
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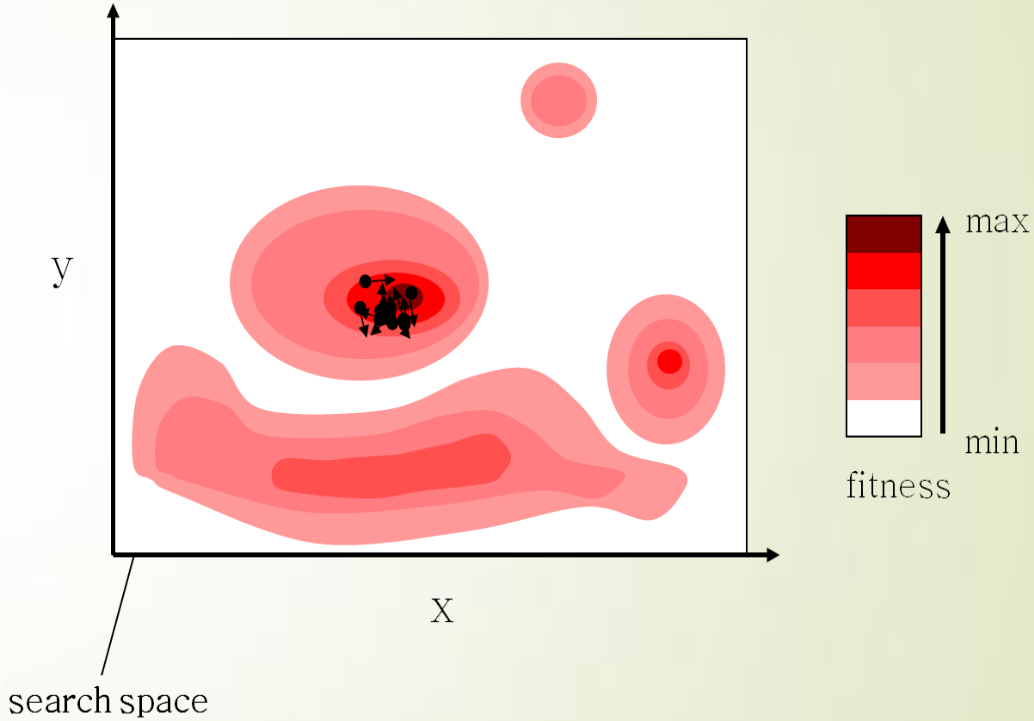
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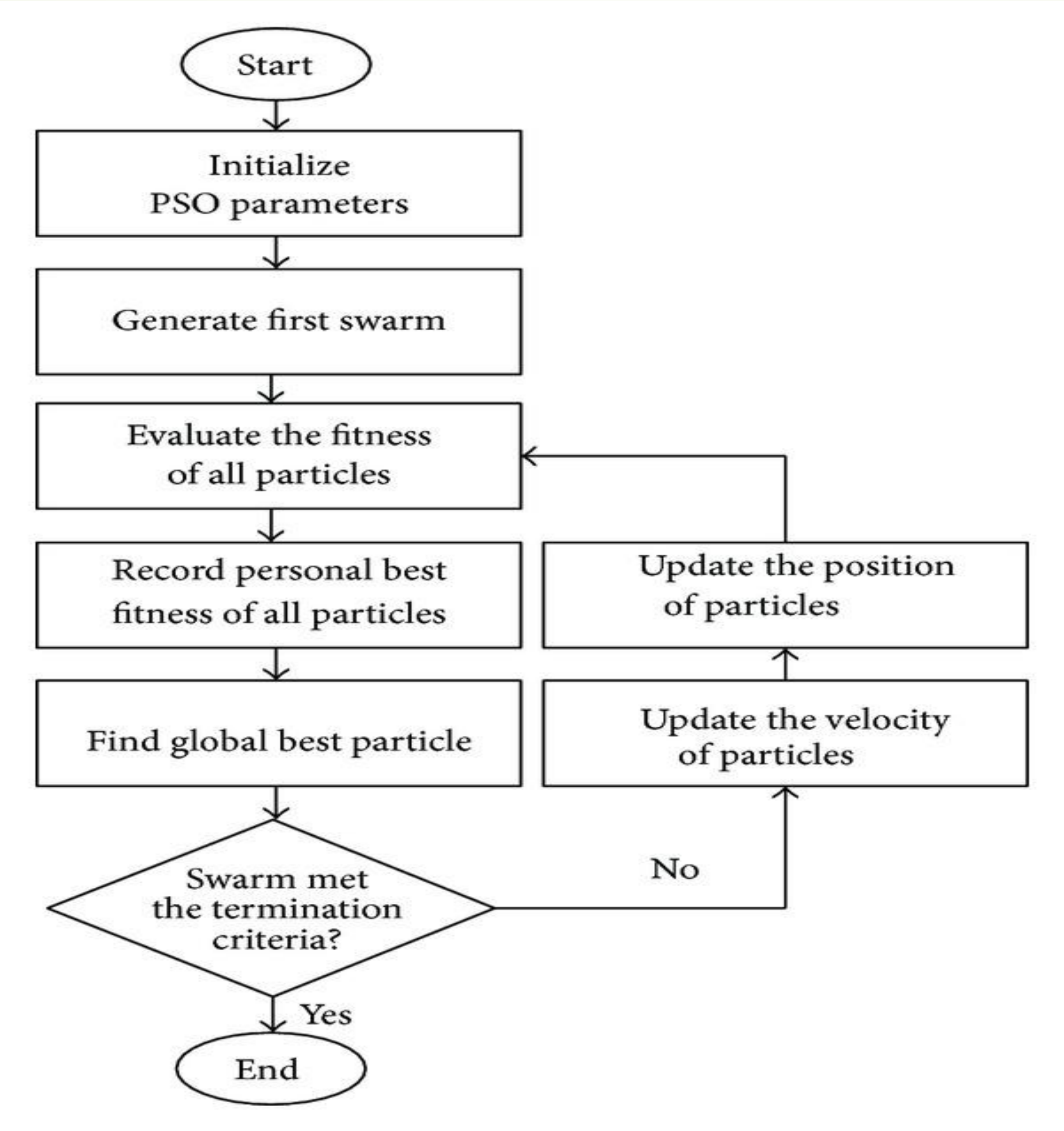
Flow chart of Algorithm



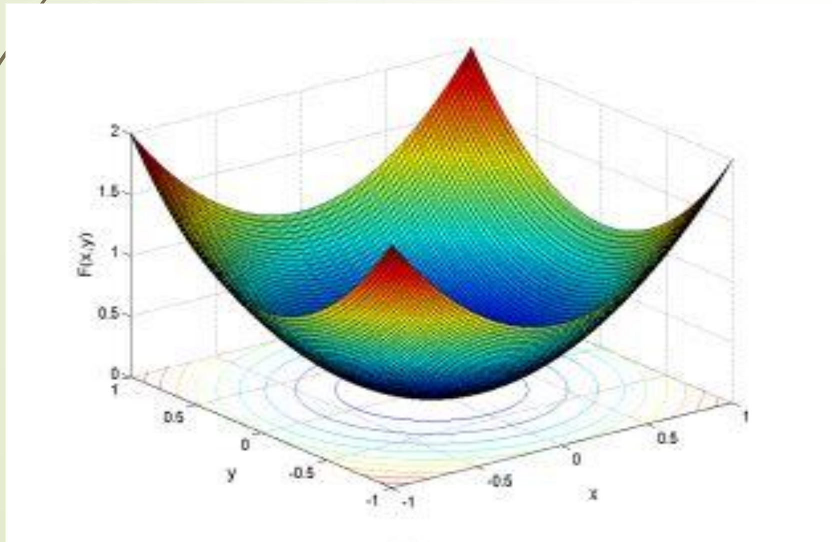
Mathematical Example and Interpretation

Fitness Function

- ✓ **De Jong Function**

$$\min F(x,y) = x^2+y^2$$

Where x and y are the dimensions of the problem. The surface and contour plot of the De Jong function is given as:



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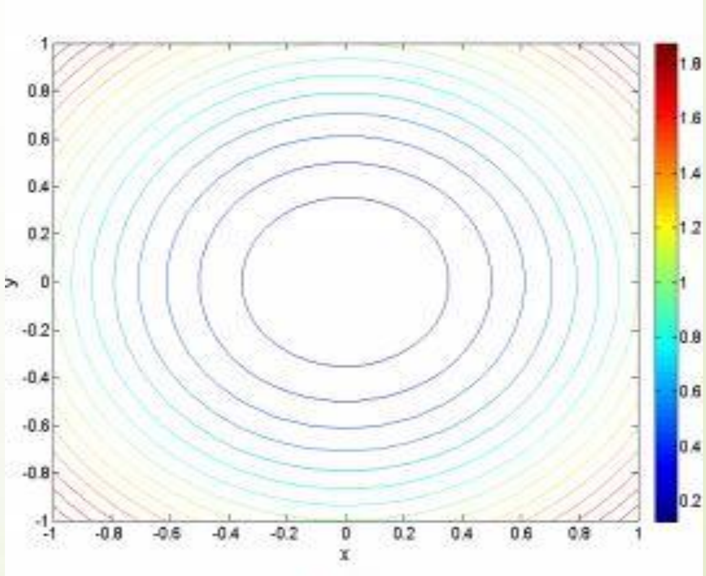


(b)

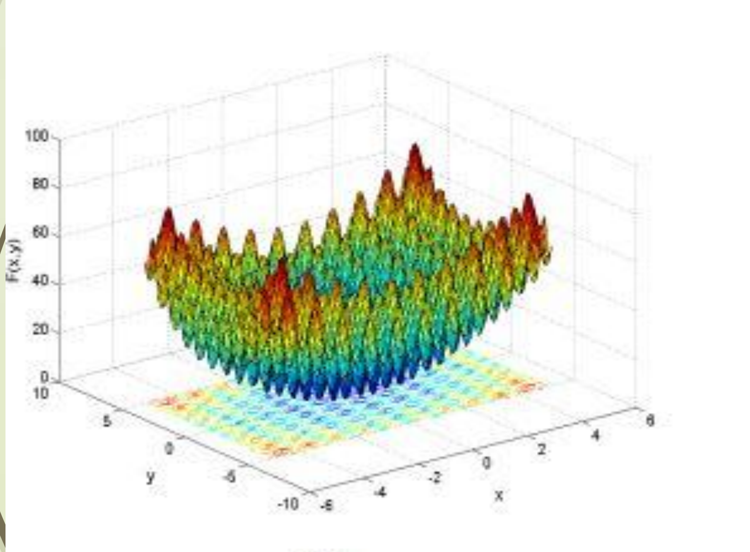
Mathematical Example and Interpretation

Contd..

Rastrigin Function

$$\sum_{t=1}^{D}(x_i - 10.\cos(2.\prod.x_i)$$

The surface and contour plot of the De Jong function is given as:



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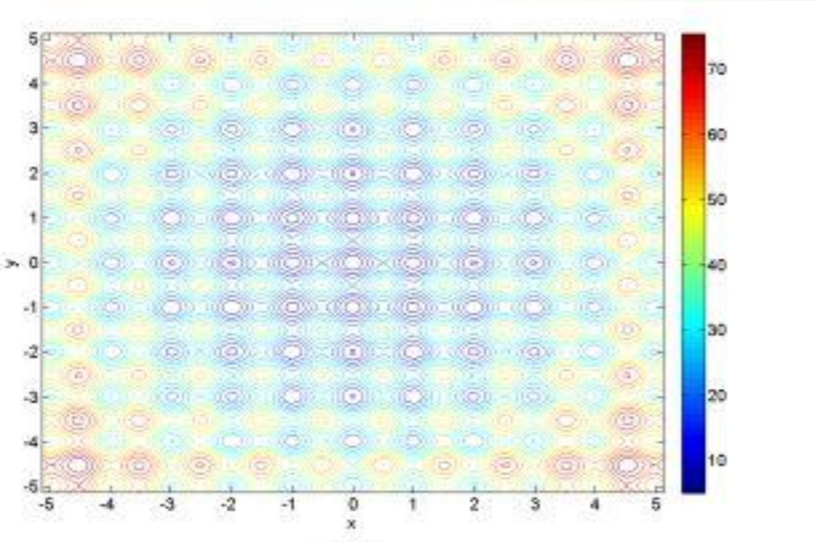

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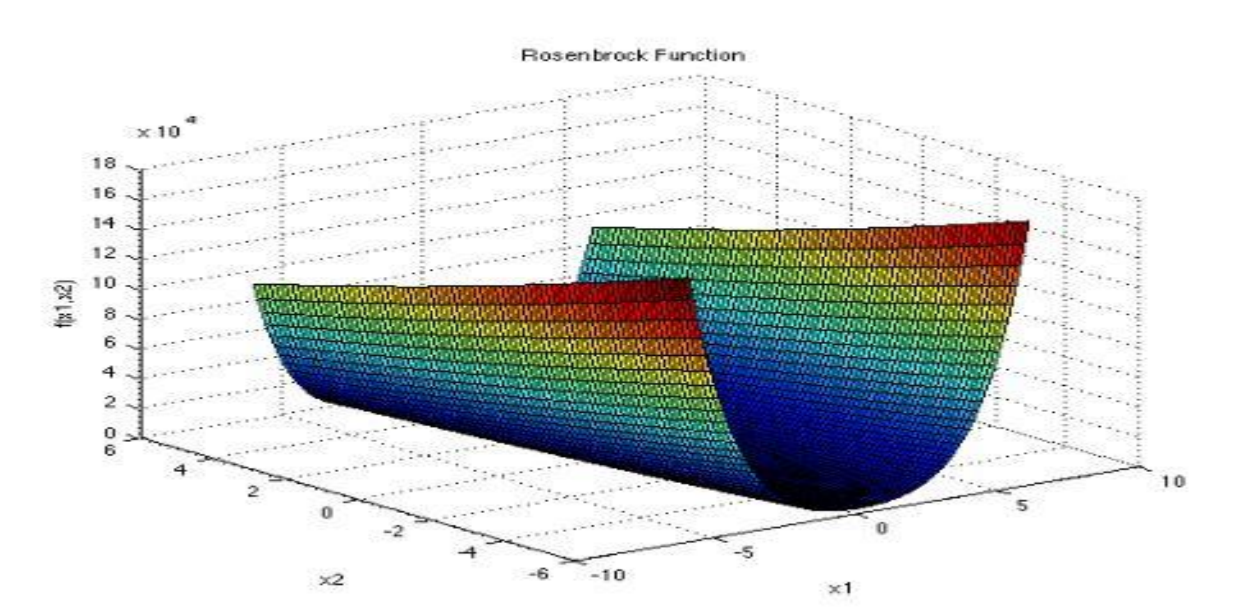
Mathematical Example and Interpretation

Contd…

Banana Function

$$f(\mathbf{x}) = \sum_{i=1}^{d-1} \left[100(x_{i+1} - x_i^2)^2 + (x_i - 1)^2\right]$$

The surface and contour plot of the Rosenbrock function or 2nd De Jong function Or valley or banana functions given as:



Mathematical Example and Interpretation

Example 1

Find the minimum of the function

$$f(x) = -x^2 + 5x + 20$$

Using PSO algorithm. Use 9 particles with initial positions

$$x_1 = -9.6, x_2 = -6, x_3 = -2.6, x_4 = -1.1,$$
$$x_5 = 0.6, x_6 = 2.3, x_7 = 2.8, x_8 = 8.3, x_9 = 10$$

Solution Choose the number of particles

$$x_1 = -9.6, x_2 = -6, x_3 = -2.6, x_4 = -1.1,$$
$$x_5 = 0.6, x_6 = 2.3, x_7 = 2.8, x_8 = 8.3, x_9 = 10$$

Evaluate the objective function

$$f_1^{\,0} = -120.16,\ f_2^{\,0} = -46,\ f_3^{\,0} = 0.24$$
$$f_4^{\,0} = 13.29,\ f_5^{\,0} = 22.64,\ f_6^{\,0} = 26.21,$$
$$f_7^{\,0} = 26.16,\ f_8^{\,0} = -7.39,\ f_9^{\,0} = -30$$

Mathematical Example and Interpretation

Contd..

Let c1=c2=1 and set initial velocities of the particles to zero.

$$v_1^{\,0} = 0, v_1^{\,0} = v_2^{\,0}, v_3^{\,0}, v_4^{\,0} = v_5^{\,0} = v_6^{\,0} = v_7^{\,0} = v_8^{\,0} = v_9^{\,0} = 0$$

Step2. Set the iteration no as t=0+1 and go to step 3

.

Step 3. Find the personal best for each particle by

$$P_{best,i}^{t+1} = \begin{cases} P_{best,i}^{t} & \text{if } f_i^{t+1} > P_{best,i}^{t} \\ x_i^{t+1} & \text{if } f_i^{t+1} \le P_{best,i}^{t} \end{cases}$$

So

$$P^1{}_{best,1} = -9.6, P^1{}_{best,2} = -6, P^1{}_{best,3} = -2.6$$

$$P^1{}_{best,4} = -1.1, P^1{}_{best,5} = 0.6, P^1{}_{best,6} = 2.3$$

$$P^1{}_{best,7} = 2.8, P^1{}_{best,8} = 8.3, P^1{}_{best,9} = 10$$

Mathematical Example and Interpretation

Contd..

Step 4: Gbest =max(Pbest) so gbest =(2.3).

Step 5: updating the velocities of the particle by considering the value of random numbers r1 = 0.213, r2= 0.876, c1=c2=1, w=1.

$$v_i^{t+1} = v_i^t + c_1 r_1^t [P_{best,i}^t - x_i^t] + c_2 r_2^t [G_{best}^t - x_i^t];\ i = 1, \dots ,9.$$

$$v_1^{\ 1} = 0 + 0.213(-9.6 + 9.6) + 0.876(2.3 + 9.6) = 10.4244$$

$$v_2^{\ 1} = 7.2708, v_3^{\ 1} = 4.2924, v_5^{\ 1} = 1.4892, v_6^{\ 1} = 0, v_7^{\ 1} = -0.4380, v_8^{\ 1} = 5.256, v_9^{\ 1} = -6.7452$$

Step 6: update the values of positions as well

$$x_i^{t+1} = x_i^t + v_i^{t+1}$$

Mathematical Example and Interpretation

Contd..

So

$$x_1^{\;1} = 0.8244,\ x_2^{\;1} = 1.2708,\ x_3^{\;1} = 1.6924$$
$$x_4^{\;1} = 1.8784,\ x_5^{\;1} = 2.0892,\ x_6^{\;1} = 2.3$$
$$x_7^{\;1} = 2.362,\ x_8^{\;1} = 3.044,\ x_9^{\;1} = 3.2548$$

Step7: Find the objective function values of

$$f_1^{\;1} = 23.4424,\ f_2^{\;1} = 24.739,\ f_3^{\;1} = 25.5978$$
$$f_4^{\;1} = 25.8636,\ f_5^{\;1} = 26.0812,\ f_6^{\;1} = 26.21$$
$$f_7^{\;1} = 26.231,\ f_8^{\;1} = 25.9541,\ f_9^{\;1} = 25.6803$$

Step 8: Stopping criteria

if the terminal rule is satisfied , go to step 2.
Otherwise stop the iteration and output the results.

Contd..

- **Step2.** Set the iteration no as t=1+1 =2   and go to step 3

Step 3. Find the personal best for each particle by

$$P^2{}_{best,1} = 0.8244, P^2{}_{best,2} = 1.2708, P^2{}_{best,3} = 1.6924$$
$$P^2{}_{best,4} = 1.87884, P^2{}_{best,5} = 2.0892, P^2{}_{best,6} = 2.3$$
$$P^2{}_{best,7} = 2.362, P^2{}_{best,8} = 3.044, P^2{}_{best,9} = 3.2548$$

Step 4: find the global best

$$G_{best} = 2.362$$

Step 5: by considering the random numbers in range (0,1) as

$$r_1{}^2 = 0.113, r_2{}^2 = 0.706$$

Contd..

- Find the velocities of the particles :

$$v_i^{t+1} = v_i^t + c_1 r_1^t [P_{best,i}^t - x_i^t] + c_2 r_2^t [G_{best}^t - x_i^t];\ i = 1, \ldots, 9.$$

$$v_1^{\ 2} = 11.5099, v_2^{\ 2} = 8.0412, v_3^{\ 2} = 4.7651, v_4^{\ 2} = 3.3198$$

$$v_5^{\ 2} = 1.6818, v_6^{\ 2} = 0.0438, v_7^{\ 2} = -0.4380, v_8^{\ 2} = -5.7375, v_9^{\ 2} = -7.3755$$

Step 6: update the values of positions as well

$$x_1^{\ 2} = 12.3343, x_2^{\ 2} = 9.312, x_3^{\ 2} = 6.4575$$

$$x_4^{\ 2} = 5.1982, x_5^{\ 2} = 3.7710, x_6^{\ 1} = 2.3438$$

$$x_7^{\ 2} = 1.9240, x_8^{\ 2} = -2.6935, x_9^{\ 2} = -4.12078$$

Contd…

- **Step7:** **Find the objective function values of**

$$f_1^2 = -70.4644,\ f_2^2 = -20.1532,\ f_3^2 = 10.5882$$
$$f_4^2 = 18.9696,\ f_5^2 = 24.6346,\ f_6^2 = 26.2256$$
$$f_7^2 = 25.9182,\ f_8^2 = -0.7224,\ f_9^2 = -17.5839$$

Step 8: Stopping criteria

if the terminal rule is satisfied , go to step 2.
Otherwise stop the iteration and output the results

Contd..

- **Step2.** Set the iteration no as t=1+2 =3 and go to step 3

Step 3. Find the personal best for each particle by

$$P^{3}{}_{best,1} = 0.8244, P^{3}{}_{best,2} = 1.2708, P^{3}{}_{best,3} = 1.6924$$
$$P^{3}{}_{best,4} = 1.87884, P^{3}{}_{best,5} = 2.0892, P^{3}{}_{best,6} = 2.3$$
$$P^{3}{}_{best,7} = 2.362, P^{3}{}_{best,8} = 3.044, P^{3}{}_{best,9} = 3.2548$$

Step 4: find the global best

$$G_{best} = 2.362$$

Step 5: by considering the random numbers in range (0,1) as

$$r_1^{\,3} = 0.178, r_2^{\,3} = 0.507$$

Find the velocities of the particles

$$v_1^3 = 4.4052, v_2^3 = 3.0862, v_3^3 = 1.8405, v_4^3 = 1.2909$$
$$v_5^3 = 0.6681, v_6^3 = 0.053, v_7^3 = -0.1380, v_8^3 = -2.1531, v_9^3 = -2.7759$$

Step 6: update the values of positions as well

$$x_1^3 = 16.7395, x_2^3 = 12.3982, x_3^3 = 8.298$$
$$x_4^3 = 6.4862, x_5^3 = 4.4391, x_6^3 = 2.3968$$
$$x_7^3 = 1.786, x_8^3 = -4.8466, x_9^3 = -6.8967$$

Step7: Find the objective function values of

Contd..

$$f_1^{\ 3} = -176.5145,\ f_2^{\ 3} = -71.7244,\ f_3^{\ 3} = -7.3673$$
$$f_4^{\ 3} = 10.3367,\ f_5^{\ 3} = 22.49,\ f_6^{\ 3} = 26..2393$$
$$f_7^{\ 3} = 25.7402,\ f_8^{\ 3} = -27.7222,\ f_9^{\ 3} = -62.0471$$

Step 8: Stopping criteria

if the terminal rule is satisfied , go to step 2.
Otherwise stop the iteration and output the results

Mathematical Example and Interpretation

Example

Iteration First:

Fitness Function :De Jong function $\min F(x,y) = x^2 + y^2$

Where x and y are the dimensions of the problem , the velocities of all the particles are initialized to zero and inertia **(W) = 0.3**, and the value of the cognitive and social constants are

C1= 2 and C2 =2. The **initial best solutions** of all the particles are set to **1000**

P1 fitness value = $1^2 + 1^2 = 2$

TABLE 1: Initial positions, velocity, and best positions of all particles.

Particle No.	Initial Positions		Velocity		Best Solution	Best Position		Fitness Value
	x	y	x	y		x	y	
P_1	1	1	0	0	1000	-	-	2
P_2	-1	1	0	0	1000	-	-	2
P_3	0.5	-0.5	0	0	1000	-	-	0.5
P_4	1	-1	0	0	1000	-	-	2
P_5	0.25	0.25	0	0	1000	-	-	0.125

Mathematical Example and Interpretation

Example

Iteration 2nd:

TABLE 2: The positions, velocity and best positions of all particles after the first iteration.

Particle No.	Initial Positions		Velocity		Best Solution	Best Position		Fitness Value
	x	y	x	y		x	y	
P_1	1	1	-0.75	-0.75	2	1	1	2
P_2	-1	1	1.25	-0.75	2	-1	1	2
P_3	0.5	-0.5	-0.25	0.75	0.5	0.5	-0.5	0.5
P_4	1	-1	-0.75	1.25	2	1	-1	2
P_5	0.25	0.25	0	0	0.125	0.25	0.25	0.125

Mathematical Example and Interpretation

Example

Iteration 3rd:

TABLE 3: The positions, velocity and best positions of all particles after the second iteration.

Particle No.	Initial Positions		Velocity		Best Solution	Best Position		Fitness Value
	x	y	x	y		x	y	
P_1	0.25	0.25	-0.3750	-0.3750	2	1	1	0.125
P_2	0.25	0.25	0.6250	-0.3750	2	-1	1	0.125
P_3	0.25	0.25	-0.1250	0.3750	0.5	0.5	-0.5	0.125
P_4	0.25	0.25	-0.3750	0.6250	2	1	-1	0.125
P_5	0.25	0.25	0	0	0.125	0.25	0.25	0.125

Mathematical Example and Interpretation

Example

Iteration 3rd:

TABLE 3: The positions, velocity and best positions of all particles after the second iteration.

Particle No.	Initial Positions		Velocity		Best Solution	Best Position		Fitness Value
	x	y	x	y		x	y	
P_1	0.25	0.25	-0.3750	-0.3750	2	1	1	0.125
P_2	0.25	0.25	0.6250	-0.3750	2	-1	1	0.125
P_3	0.25	0.25	-0.1250	0.3750	0.5	0.5	-0.5	0.125
P_4	0.25	0.25	-0.3750	0.6250	2	1	-1	0.125
P_5	0.25	0.25	0	0	0.125	0.25	0.25	0.125

Psuedocode

1. P=particle initialization();
2. For I =1 to max
   3 for each particle p in P do
   fp=f(p)
4. If fp is better than f(pbest)
   pbest =p;
5. end
6. end
7. gbest = best p in P.
8. for each particle p in P do
9. $$\mathbf{v}_i^{t+1} = \underbrace{\mathbf{v}_i^t}_{inertia} + \underbrace{c_1 \mathbf{U}_1^t(\mathbf{pb}_i^t - \mathbf{p}_i^t)}_{personal\ influence} + \underbrace{c_2 \mathbf{U}_2^t(\mathbf{gb}^t - \mathbf{p}_i^t)}_{social\ influence}$$
10. $$\mathbf{p}_i^{t+1} = \mathbf{p}_i^t + \mathbf{v}_i^{t+1}$$
11. end
12. end

DataSet

Advantages and Disadvantages of PSO

Advantages

- ✓ Insensitive to scaling of design variables.
- ✓ Simple implementation.
- ✓ Easily parallelized for concurrent processing.
- ✓ Derivative free.
- ✓ Very few algorithm parameters.
- ✓ Very efficient global search algorithm.

Disadvantages

- ✓ **Slow convergence in refined search stage (weak local search ability).**