Classical Electrodynamics

Lecture Series (Course MSPHY 201)

By

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Lecture 01:

Introduction:

If we look around, all kinds of processes that are happening can be traced to be due to four fundamental interactions. These interactions are strong and weak, dominant in subatomic world and electromagnetic plus gravitational which dictate almost all phenomena at macroscopic level. Most of the phenomena that we directly interact are electromagnetic in origin. Ordinary pull or push, normal reaction on a book resting on a table, friction experienced by a rolling object, tension in a rope that pulls a cart, forces generated by our muscles are all electromagnetic in origin. Understanding of electromagnetic interactions between current and charge distributions and their behaviour in the presence of fields is technically called Classical Electrodynamics.

At deeper level classical electrodynamics is unified theory of electricity, magnetism and optics which was shaped by Carl Maxwell through his beautiful equations famously called Maxwell's equations of electrodynamics. The impact of this theory was immense on future developments in theoretical physics. Modern theories of physics like QFT, QCD and even string theories (An attempt of a unified theory of everything) are actually extensions of classical electrodynamics.

We start with brief introduction of electrostatics. Initially electrostatics, magnetism and optics where separate subjects besides thermodynamics and mechanics. It was Maxwell's genius that we where able to crack symmetry and could understand nature in a better way.

Electrostatics (A brief introduction) : We begin with Coulomb's law, experiments at classical level reveal that the force between static charge distributions can be understood if we assume force between two charged particles is directly proportional to product of charges between the particles and inversely related with square of displacement between them. This statement is called Coulomb's law. Further it is also assumed that forces occur pairwise (superposition principle). Mathematically we express this fact through the following equation

$$F(r) = \frac{qq'(r-r')}{4\pi\epsilon_0 |r-r'|^3}$$

$$= \frac{-qq'}{4\pi\epsilon_0}\nabla\left(\frac{1}{|r-r'|}\right)$$

Where 'q' and 'q'' are two point charges separated by displacement vector $(r-r')$. The above equation is expressed interims of system of international units. The main draw back of the above expression of force is that the interaction is instantaneous and is action at a distance statement. This feature is ugly in any kind of theoretical structure as it implies instantaneous transfer of information (something that seems absurd).

The problem of action at a distance is addressed by introducing the concept of field. Fields fill empty space between the charges and make interaction between two bodies local .i.e the test charges are locally influenced by fields created by source charges. Initially Fields where introduced to make interaction local as mathematical concepts but with progress of our understanding fields turned out to be physical entities expressing physical reality of our universe. Today we understand everything interns of fields, even mass is due to a field called Higgs.

Electrostatic interaction is expressed interims of a vector field called electric field as

$$
\begin{aligned}
E(r) &= \frac{q'(r-r')}{4\pi\epsilon_0 |r-r'|^3} \\
&= \frac{-q'}{4\pi\epsilon_0} \nabla\left(\frac{1}{|r-r'|}\right) = \frac{-q'}{4\pi\epsilon_0} \nabla'\left(\frac{1}{|r-r'|}\right)
\end{aligned} \tag{1}
$$

In the presence of several source charges, the net field is vector sum of fields due to individual charges. We write

$$
E(r) = \sum_i \frac{q_i'(r-r_i')}{4\pi\epsilon_0 |r-r_i'|^3} \tag{2}
$$

Incase of continuous charge distribution, we introduce electric charge density "ρ" located at r' within volume V'. Then the field for this kind of distribution is defined as

$$
\begin{aligned}
E(r) &= \frac{1}{4\pi\epsilon_0} \int_{V'} d^3r' \rho(r') \frac{r-r'}{|r-r'|^3} \\
&= -\frac{1}{4\pi\epsilon_0} \int_{V'} d^3r' \rho(r') \nabla\left(\frac{1}{r-r'}\right) \\
&= -\frac{1}{4\pi\epsilon_0} \nabla \int_{V'} d^3r' \frac{\rho(r')}{r-r'}
\end{aligned} \tag{3}
$$

The above expression for any distribution of charges including discrete charges for which we use Dirac delta distribution

$$
\rho(r') = \sum_i q_i' \delta(r'-r_i') \tag{4}
$$

If we insert equation (4) in equation (3), we recover equation (2). Now for the general expression of electric field defined through equation (3), we take divergence and use property of Dirac delta function, we obtain

$$\nabla . E(r) = \nabla . \frac{1}{4\pi\epsilon_0} \int_{V'} d^3r' \rho(r') \frac{r - r'}{|r - r'|^3}$$

$$= -\frac{1}{4\pi\epsilon_0} \int_{V'} d^3r' \rho(r') \nabla . \nabla \left(\frac{1}{r - r'} \right)$$

$$= -\frac{1}{4\pi\epsilon_0} \int_{V'} d^3r' \rho(r') \nabla^2 \left(\frac{1}{r - r'} \right)$$

$$= -\frac{1}{\epsilon_0} \int_{V'} d^3r' \rho(r') \delta(r - r')$$

$$\nabla . E(r) = \frac{\rho(r)}{\epsilon_0} \tag{5}$$

Equation (5) is differential form of Gauss law in electrostatics. Since curl of divergence is zero for any field in R^3 (three dimensional), we can easily write down

$$\nabla X E(r) = 0 \tag{6}$$

Thus we prove that electrostatic field is irrotational. A curl less field represents conservative nature of force which defines the field.

Electrostatics can completely be described by two vector partial differential equations (5) and (6). These represent four scaler equations. Since curl of electric field vanishes, because in equation (3), we expressed electric filed interns of divergence of scaler function. We name this scaler function as scaler potential and express it by **V** Hence we write

$$E(r) = -\nabla V \tag{7}$$

Inserting equation (7) in (6), we obtain

$$\nabla^2 V = \frac{-\rho(r)}{\epsilon_0} \tag{8}$$

This second order differential equation turns out to be master equation of electrostatics called Poisson's equation. In regions devoid of charges, this equation reduces to Laplace equation given as

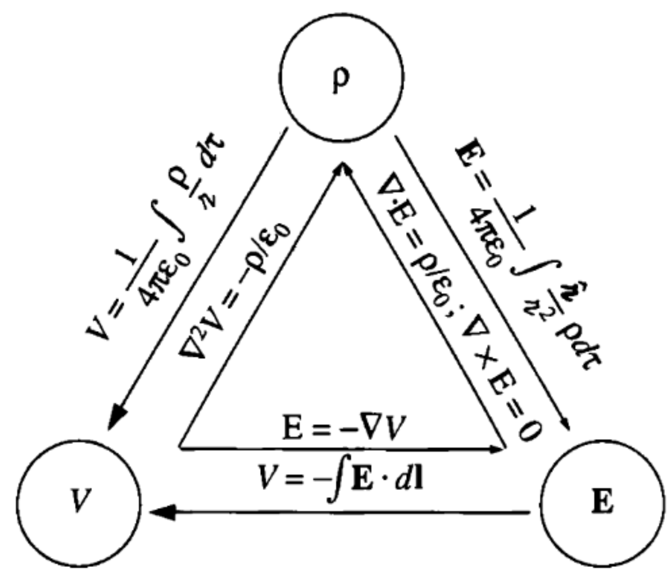
$$\nabla^2 V = 0 \tag{9}$$

In the next lecture we will discuss electrostatic boundary conditions.

Lecture 02: Electrostatic Boundary conditions

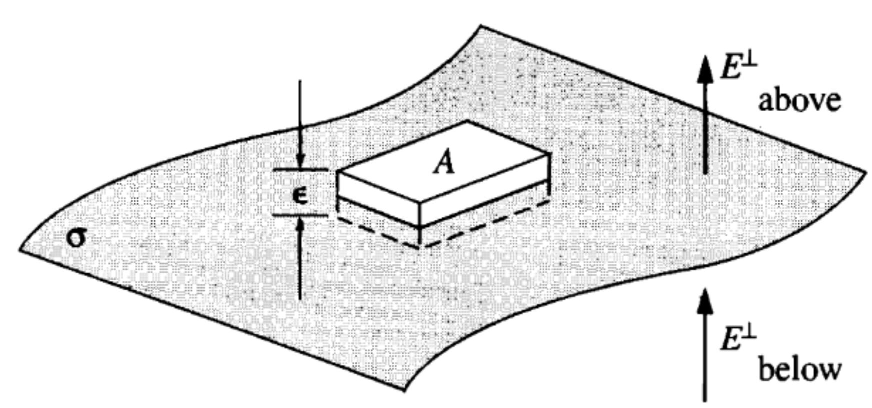
Normally the problem of electrostatics is that we have some given charge distribution $\rho(r)$ and our job is to calculate Electric filed due to this static distribution. The approach is to first look for symmetry that might allow to implement Gauss's law. If we can not figure out a simple symmetric Gaussian surface then we generally adopt procedure to calculate potential first as intermediate step.

Thus there are three fundamental quantities $\rho(r)$, **E** and **V.** There are six relations that relate these quantities depicted by this triangle which originate from consideration of Coulomb's law and the principle of superposition - the fundamental building blocks of electrostatics.



In the light of relations described in the above triangle it is very easy to obtain variation of **E** field across and parallel to a boundary sprayed with surface charge density σ as shown in the figure given below. We draw a pillbox extending just across the surface. Now we write

$$\oint_s E \, . \, da = \frac{Q}{\epsilon_0} = \frac{\sigma A}{\epsilon_0} \qquad \text{...............} \qquad (10)$$



"A" is the area of pillbox lid. Since the sides of pillbox contribute nothing to the flux and in the limit as thickness $\epsilon \to 0$, we retain

$$E^{\perp}_{above} - E^{\perp}_{below} = \frac{\sigma}{\epsilon_0} \qquad \text{..................} \qquad (11)$$

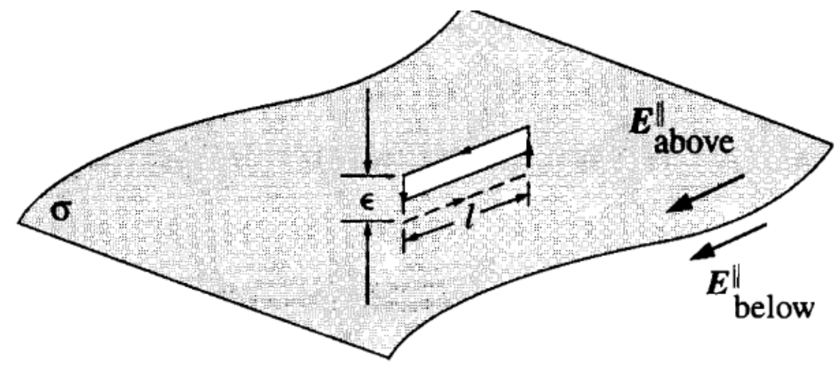
From equation (11) we conclude that the normal component of "**E**" is discontinuous by an amount $\frac{\sigma}{\epsilon_0}$ at any boundary. In case there are no surface charges, then **E** field would be continuous.

The tangential component will always be continuous due to the fact that **E**- field is conservative and follows

$$\oint E \,.\, dl = 0 \tag{12}$$

To prove the continuity of tangential component of "E", draw a rectangular loop of very small width tending to zero as shownin figure. Implementing equation (12), we obtain

$E^{||}_{above} = E^{||}_{below}$ .... (13)



The boundary conditions on **"E"** can be combined into a single formula.

$E_{above} - E_{below} = \dfrac{\sigma \hat{n}}{\epsilon_0}$ ....(14)

Potential difference can be written as

$$V_{above} - V_{below} = -\int_a^b E \,.\, dl \tag{15}$$

The internal will vanish as path length shrinks to zero, thus

$$V_{above} = V_{below} \tag{16}$$

Gradient of V will be discontinuous in accordance with equation (10), we write

$$\nabla V_{above} - \nabla V_{below} = -\frac{\sigma \hat{n}}{\epsilon_0} \tag{17}$$

Thus what we have understood so far is that if a static charge distribution is given, we may discover some symmetry in the distribution which would allow us to exploit Gauss law and in case we could not solve the problem we will try to calculate potential first and workout field. We do come across problems where even this step is challenging for instance problems involving conductor ρ itself may not be known in advance as for conductors charge can freely move around so one can only control net charge of a conductor. In these problems the differential form of potential (Poisson equation) together with appropriate boundary conditions allows us to evaluate the potential field and thus address complicated problems of electrostatics.

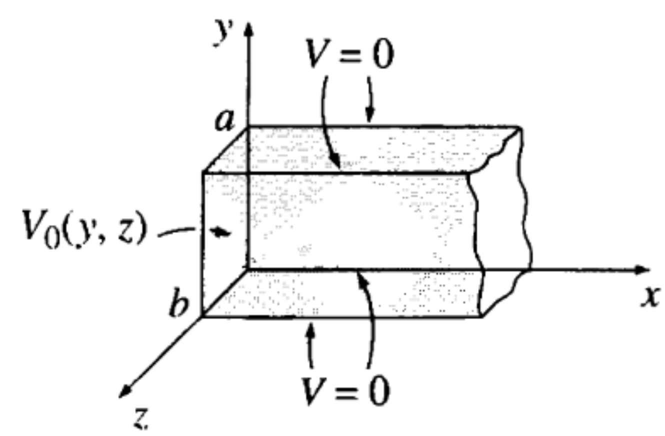
Laplace equation:

If we distribute charge in some region but are interested to look at potential where no charges are present, Poisson equation reduces to Laplace equation given by

$$\nabla^2 V = 0 \ : \tag{19}$$

Problem 1: (D J Grifith- 3rd edition)

An infinitely log rectangular metal pipe (sides a and b) is grounded, but one end, at x=0, is maintained at a specified potential $V_0(x, y)$, as indicated in figure. Find the potential inside the pipe.



Solution :

This is a three dimensional problem with Cartesian symmetry, therefore we attempt to solve the problem is Cartesian coordinates. Writing Laplace equation in Cartesian form, we have

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0 \tag{20}$$

The given problem is subjected to following boundary conditions

i. $V = 0$ When $y = 0$

ii. $V = 0$ When $y = a$

iii. $V = 0$ When $z = 0$

iv. $V = 0$ When $z = b$

v. $V \to 0$ When $x \to \inf$

vi. $V = V_0(y, z)$ When $x = 0$

In order to solve the given problem we look for solutions that satisfy the product assumption

$$V(x, y, z) = X(x)Y(y)Z(z) \tag{21}$$

Substituting (21) in (20) and dividing by V, we get

$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2}+\frac{1}{Y}\frac{\partial^2 Y}{\partial y^2}+\frac{1}{Z}\frac{\partial^2 Z}{\partial z^2}=0 \tag{22}$$

All terms in the above equation are functions of one independent variable. Therefore the only way the above equation makes sense is that each term is individually is equal to some constant and all the three constants sum up to zero. We say,

$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2}=C_1,\frac{1}{Y}\frac{\partial^2 Y}{\partial y^2}=C_2,\frac{1}{Z}\frac{\partial^2 Z}{\partial z^2}=C_3 \tag{23}$$

With

$$C_1+C_2+C_3=0. \tag{24}$$

Now the problem is that one may think that we can choose these constants arbitrarily positive and negative to satisfy equation (24) but one has to be careful because our solution has to satisfy boundary conditions, therefore we try many combinations and it turns out that following choice is proper recipe

$C_2=-k^2, C_3=-l^2$ which fixes $C_1=k^2+l^2$ and hence

$$\frac{\partial^2 X}{\partial x^2}=(k^2+l^2)X,\frac{\partial^2 Y}{\partial y^2}=-k^2Y,\frac{1}{Z}\frac{\partial^2 Z}{\partial z^2}=-l^2Z \tag{25}$$

With the help of separation of variables, we have converted three dimensional partial differential equation into three one dimensional ordinary differential equations. Solving ordinary differential equations is straight forward and procedures are well understood.

Solving three equations in (25) independently we have following solutions

$$X(x)=Ae^{\sqrt{k^2+l^2}}+Be^{-\sqrt{k^2+l^2}}x$$

$$Y(y)=Csinky+Dcosky$$

$$Z(z)=Esinlz+Fcoslz$$

Implementing boundary conditions from (i) to (iv), requires $A=0, D=0, F=0$ and $k=\frac{n\pi}{a}$ and $l=\frac{m\pi}{b}$, where n and m are positive integers. Combing the remaining constants, we are left with

$$V(x,y,z)=Ce^{-\pi\sqrt{(\frac{n}{a})^2+(\frac{m}{b})^2}x}sin\left(\frac{n\pi y}{a}\right)sin\left(\frac{m\pi z}{b}\right) \tag{26}$$

The above solution meets all boundary conditions except last one. General solution can be expressed as double sum over n, m as

$$V(x,y,z) = \sum_{n}\sum_{m} C_{n,m} e^{-\pi\sqrt{(\frac{n}{a})^2 + (\frac{m}{b})^2}x} sin\left(\frac{n\pi y}{a}\right) sin\left(\frac{m\pi z}{b}\right) \tag{27}$$

Coefficients $C_{n,m}$ can be determined by multiplying above equation (27) by $sin\left(\frac{n'\pi y}{a}\right) sin\left(\frac{m'\pi z}{b}\right)$, and implementing last boundary condition. On integrating both sides we get

$$\sum_{n}\sum_{m} C_{n,m}\int_0^a sin\left(\frac{n\pi y}{a}\right) sin\left(\frac{n'\pi y}{a}\right) dy \int_0^b sin\left(\frac{m\pi z}{b}\right) sin\left(\frac{m'\pi z}{b}\right) dz$$

$$= \int_0^a\int_0^b V_0(y,z) sin\left(\frac{n'\pi y}{a}\right) sin\left(\frac{m'\pi z}{b}\right) dydz \tag{28}$$

Evaluating, we obtain

$$C_{n,m} = \frac{4}{ab}\int_0^a\int_0^b V_0(y,z) sin\left(\frac{n\pi y}{a}\right) sin\left(\frac{m\pi z}{b}\right) dydz \tag{29}$$

Further, the above integral takes zero value for even n and m and for odd values integrals converge to $\frac{16V_0}{\pi^2 nm}$, thus we obtain final solution as

$$V(x,y,z) = \frac{16V_0}{\pi^2}\sum_{n,m=1,3,5..}\frac{1}{nm} e^{-\pi\sqrt{(\frac{n}{a})^2 + (\frac{m}{b})^2}x} sin\left(\frac{n\pi y}{a}\right) sin\left(\frac{m\pi z}{b}\right) \tag{30}$$

Successive terms decrease rapidly, a good approximation can be obtained by keeping first few terms only.

Home Assignment 1:

Write Mathematica code for visualisation of equation (30). Compare approximate solution.

Problems with spherical symmetry:

Laplace equation spherical coordinates takes the form

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right)+\frac{1}{r^2 sin\theta}\frac{\partial}{\partial \theta}\left(sin\theta\frac{\partial V}{\partial \theta}\right)+\frac{1}{r^2 sin^2\theta}\left(\frac{\partial^2 V}{\partial^2 \phi}\right)=0 \tag{31}$$

If we assume problem to be having azimuthal symmetry ie potential is independent of ϕ coordinate, the above equation reduces to

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right)+\frac{1}{r^2 sin\theta}\frac{\partial}{\partial \theta}\left(sin\theta\frac{\partial V}{\partial \theta}\right)=0 \tag{32}$$

We again follow the method of separation of variables and assume that potential takes product form

$$V(r,\theta)=R(r)\Theta(\theta) \tag{33}$$

Substituting (33) in (32), we obtain

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)+\frac{1}{\Theta sin\theta}\frac{d}{d\theta}\left(sin\theta\frac{d\Theta}{d\theta}\right)=0 \tag{34}$$

Then first term in the above equation is function of ‘r’ only and the second term is function of θ only. Therefore , they must individually be equal to some constants whose sum will vanish. For mathematical simplicity and convenance we choose

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)=l(l+1) \quad \text{and} \quad \frac{1}{\Theta sin\theta}\frac{d}{d\theta}\left(sin\theta\frac{d\Theta}{d\theta}\right)=-l(l+1) \tag{35}$$

Solving these ordinary differential equations, we have solution of the first equation as

$$R(r)=Ar^l+\frac{B}{r^{l+1}},$$

The solution of angular equation is a bit difficult. The solutions are Legendre polynomials in the variable $cos\theta$.

$$\Theta(\theta)=P_l(cos\theta) \tag{36}$$

$P_l(x)$ is most conveniently defined by the Rodrigues formula:

$$P_l(x)=\frac{1}{2^l l!}\left(\frac{d}{dx}\right)^l(x^2-1)^l \tag{37}$$

The first few Legendre polynomials are

$$\begin{array}{|lcl|}\hline P_0(x) & = & 1 \\ P_1(x) & = & x \\ P_2(x) & = & (3x^2-1)/2 \\ P_3(x) & = & (5x^3-3x)/2 \\ P_4(x) & = & (35x^4-30x^2+3)/8 \\ P_5(x) & = & (63x^5-70x^3+15x)/8 \\ \hline \end{array}$$

Rodrigues formula works only for nonnegative integer values of "l" . In case of azimuthal symmetry, most general solution of Laplace equation is

$$V(r,\theta) = \left(Ar^l + \frac{B}{r^{l+1}}\right) P_l(cos\theta) \tag{38}$$

The general solution is linear combination of separable solutions

$$V(r,\theta) = \sum_{l=0} \left(A_l r^l + \frac{B_l}{r^{l+1}}\right) P_l(cos\theta) \tag{39}$$

Problem 2: The potential $V_0(\theta)$ is specified on the surface of a hollow sphere, of radius R. Find the potential inside the sphere.

Solutions: In this case we choose $B_l = 0$ because otherwise solution would become infinite at $r = 0$, thus we are left with

$$V(r,\theta) = \sum_{l=0} \left(A_l r^l\right) P_l(cos\theta) \tag{40}$$

Now we impose that at $r = R$, the solution must match with value at boundary, so we have

$$V(R,\theta) = \sum_{l=0} \left(A_l R^l\right) P_l(cos\theta) = V_0(\theta) \tag{41}$$

Constants A_l can easily be evaluated by using following properties of Legendre polynomials.

$$\begin{aligned} \int_{-1}^{1} P_l(x) P_{l'}(x)\, dx &= \int_0^\pi P_l(\cos\theta) P_{l'}(\cos\theta) \sin\theta \, d\theta \\ &= \begin{cases} 0, & \text{if } l' \neq l, \\ \dfrac{2}{2l+1}, & \text{if } l' = l. \end{cases} \end{aligned}$$

Multiplying equation (41) by $P_{l'}(cos\theta)sin\theta$ and integrating, we get

$$A_{l'}R^{l'}\frac{2}{2l'+1} = \int_0^\pi V_0(\theta)P_{l'}(cos\theta)sin\theta d\theta$$

$$A_l = \frac{2l+1}{2R^l}\int_0^\pi V_0(\theta)P_l(cos\theta)sin\theta d\theta \tag{42}$$

Equation (40) represents solution of our problem with coefficients given by equation (42). All we need to know is potential as function of θ.

Assignment: Try solving the problem for various forms of potential as function of θ, you can begin with constant potential to see if our procedure is correct prescription.

Problem 3: The potential $V_0(\theta)$ is again specified on the surface of a sphere of radius R , but this time we are asked to find the potential outside, assuming there is no charge there.

Solution: In this case we choose A_l 's to be zero because the first term in general solution is increasing with 'r' and would get infinity as $r \to \infty$, therefore we are left with

$$V(r,\theta) = \sum_{l=0}^{\infty}\frac{B_l}{r^{l+1}}P_l(cos\theta) \tag{43}$$

Imposing the surface boundary condition, we have

$$V(R,\theta) = \sum_{l=0}^{\infty}\frac{B_l}{R^{l+1}}P_l(cos\theta) = V_0(\theta) \tag{44}$$

Again multiplying with $P_{l'}(cos\theta)sin\theta$ and integrating, we obtain.

$$\frac{B_{l'}}{R^{l'+1}}\frac{2}{2l'+1} = \int_0^\pi V_0(\theta)P_{l'}(cos\theta)sin\theta d\theta$$

$$B_l = \frac{2l+1}{2}R^{l+1}\int_0^\pi V_0(\theta)P_l(cos\theta)sin\theta d\theta \tag{45}$$

For any potential as function of θ, we can solve for potential outside the sphere by fixing constants through equation (45) and substituting in equation (43).

Assignment:

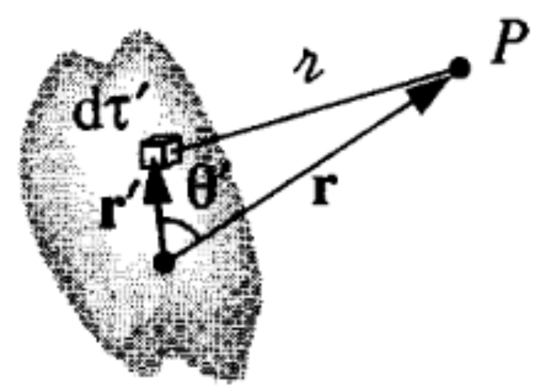
1. **The potential at the surface of a sphere (radius R) is given by $V_0 = kcos3\theta$, where k is a constant. Find the potential inside and outside the sphere, as well as the surface charge density $\sigma(\theta)$ on the sphere. (Assume there's no charge inside or outside the sphere.**

2. **Solve Laplace's equation by separation of variables in cylindrical coordinates, assuming there is no dependence on z (cylindrical symmetry).**

3. **Find the potential outside an infinitely long metal pipe, of radius R, placed at right angles to an otherwise uniform electric field E_0. Find the surface charge induced on the pipe.**

Multipole Expansion:

Consider a localised charge distribution described by charge density $\rho(r')$. Let's try to evaluate potential at any arbitrary point P located at displacement "**r**" from chosen origin "o" as shown in figure.



We can divide the distribution into infinitesimal volume elements $d\tau'$. The potential at P due to entire distribution is given by

$$V(r) = \frac{1}{4\pi\epsilon_0}\int \frac{1}{r}\rho(r')d\tau' \qquad (46)$$

Using cosine law we have

$$\mathscr{r}^2 = r^2 + (r')^2 - 2rr'\cos\theta' = r^2\left[1 + \left(\frac{r'}{r}\right)^2 - 2\left(\frac{r'}{r}\right)\cos\theta'\right],$$

$$\mathscr{r} = r\sqrt{1+\epsilon}$$

$$\epsilon \equiv \left(\frac{r'}{r}\right)\left(\frac{r'}{r} - 2\cos\theta'\right).$$

For points outside the distribution, ϵ is much less than 1, therefore we can binomial expand

$$\frac{1}{\mathscr{r}} = \frac{1}{r}(1+\epsilon)^{-1/2} = \frac{1}{r}\left(1 - \frac{1}{2}\epsilon + \frac{3}{8}\epsilon^2 - \frac{5}{16}\epsilon^3 + \ldots\right), \tag{46}$$

The above expression can now be presented in terms of Legendre polynomials,

$$\begin{aligned}\frac{1}{\mathscr{r}} &= \frac{1}{r}\Bigg[\quad \frac{1}{\mathscr{r}} = \frac{1}{r}\sum_{n=0}^{\infty}\left(\frac{r'}{r}\right)^n P_n(\cos\theta'), \quad \Bigg)^2 \left(\frac{r'}{r} - 2\cos\theta'\right)^2 \\ &\qquad - \frac{5}{16}\left(\frac{r'}{r}\right)^3 \left(\frac{r'}{r} - 2\cos\theta'\right)^3 + \ldots\Bigg] \\ &= \frac{1}{r}\left[1 + \left(\frac{r'}{r}\right)(\cos\theta') + \left(\frac{r'}{r}\right)^2 (3\cos^2\theta' - 1)/2 \right. \\ &\qquad \left. + \left(\frac{r'}{r}\right)^3 (5\cos^3\theta' - 3\cos\theta')/2 + \ldots\right].\end{aligned}$$

Now the final expression for potential can now be written as

$$\boxed{V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\sum_{n=0}^{\infty}\frac{1}{r^{(n+1)}}\int (r')^n P_n(\cos\theta')\rho(\mathbf{r}')\,d\tau',}$$

Or

$$\begin{aligned}V(\mathbf{r}) &= \frac{1}{4\pi\epsilon_0}\left[\frac{1}{r}\int \rho(\mathbf{r}')\,d\tau' + \frac{1}{r^2}\int r'\cos\theta'\rho(\mathbf{r}')\,d\tau' \right. \\ &\qquad \left. + \frac{1}{r^3}\int (r')^2\left(\frac{3}{2}\cos^2\theta' - \frac{1}{2}\right)\rho(\mathbf{r}')\,d\tau' + \ldots\right].\end{aligned} \tag{47}$$