Modern Physics:Lecture 03: Wave Particle Duality

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Outline

Historical Background

- Newton: Light as particle (corpuscles).
- Huygens: Light as wave.
- Young's double-slit experiment confirms wave nature.
- Planck and Einstein: Photons, quantized energy.
- de Broglie: Extended duality to matter.

Einstein's Photon Hypothesis

- Energy of a photon: $E = h\nu$.
- Momentum of a photon: $p = \frac{h}{\lambda}$.
- Raises question: Can particles exhibit wave-like behavior?

de Broglie's Postulate

Matter-Wave Relation

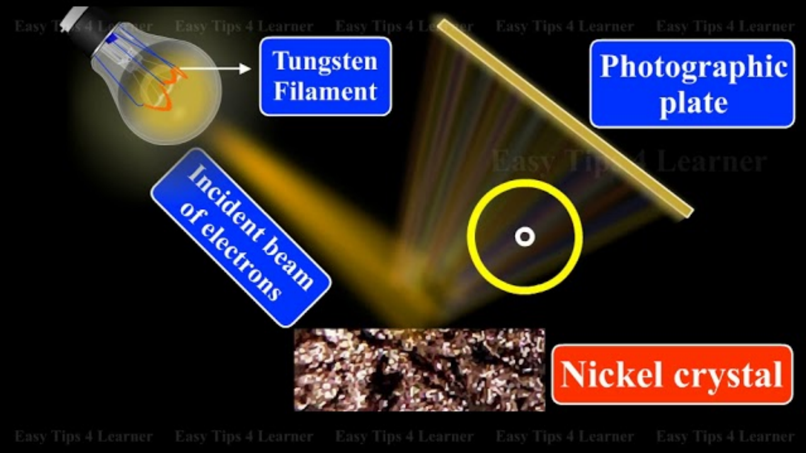
Every moving particle is associated with a wave:

$$\lambda = \frac{h}{p}$$

Justification Using Special Relativity

- Energy-momentum relation: $E^2 = p^2c^2 + m_0^2c^4$.
- de Broglie assumed wave relation $E = h\nu$ still holds.
- Combined with $p = \frac{h}{\lambda}$ gives coherent interpretation.

Experimental Evidences



Phase Velocity of de Broglie Waves

- Consider a free particle of mass m and momentum p.
- Associated de Broglie wavelength: $\lambda = \frac{h}{p}$.
- Angular frequency: $\omega = \frac{E}{\hbar}$.
- Wavenumber: $k = \frac{p}{\hbar}$.
- Phase velocity: $v_p = \frac{\omega}{k} = \frac{E}{p}$.

Problem with Phase Velocity

- For a non-relativistic particle:

$$E = \frac{p^2}{2m} \Rightarrow v_p = \frac{E}{p} = \frac{p}{2m} = \frac{v}{2}$$

- Phase velocity is only half the actual particle velocity.
- For a relativistic particle: $E = \gamma mc^2$, $p = \gamma mv$:

$$v_p = \frac{E}{p} = \frac{\gamma mc^2}{\gamma mv} = \frac{c^2}{v} > c$$

- This exceeds the speed of light c—non-physical.

Wave Packet Representation

- Construct a localized particle using a superposition of plane waves:

$$\psi(x,t) = \int A(k) e^{i(kx-\omega t)} dk$$

- The resulting wave packet has a group of closely spaced wave numbers around k_0.

Definition of Group Velocity

- Group velocity is defined as:

$$v_g = \frac{d\omega}{dk}$$

- It represents the velocity of the envelope of the wave packet.

Group Velocity for Free Particle (Non-Relativistic)

- Energy: $E = \frac{p^2}{2m} = \frac{\hbar^2 k^2}{2m}$.
- Then $\omega = \frac{E}{\hbar} = \frac{\hbar k^2}{2m}$.
- Group velocity:

$$v_g = \frac{d\omega}{dk} = \frac{\hbar k}{m}$$

- But $p = \hbar k$, so:

$$v_g = \frac{p}{m} = v$$

- Hence, group velocity equals particle velocity.

Group Velocity for Free Particle (Relativistic)

- Relativistic energy: $E^2 = p^2c^2 + m^2c^4$.
- $\omega = \frac{E}{\hbar}$, $k = \frac{p}{\hbar}$.
- Use chain rule:

$$v_g = \frac{d\omega}{dk} = \frac{dE/\hbar}{dp/\hbar} = \frac{dE}{dp}$$

- But:

$$\frac{dE}{dp} = \frac{pc^2}{E} = v$$

- Again, group velocity equals particle velocity.

Resolution of Paradox

- Phase velocity can be greater than c—not an issue, no energy or information travels at v_p.
- Group velocity corresponds to particle motion.
- It is subluminal and physically meaningful.

Conclusion

- Phase velocity fails to describe real particle motion.
- Superposition leads to wave packets.
- Group velocity correctly describes particle velocity in both non-relativistic and relativistic regimes.

Uncertainty Principle from wave packet consideration

- The wave nature of matter implies that particles are described by wave packets.
- Localization of a particle in space requires a spread of wave numbers.
- This leads naturally to the uncertainty principle.

Wave Packet Construction

- A wave packet is a superposition of plane waves:

$$\psi(x) = \int_{-\infty}^{\infty} A(k) e^{ikx} dk$$

- Choose $A(k)$ as a Gaussian:

$$A(k) = A_0 e^{-\frac{(k-k_0)^2}{2(\Delta k)^2}}$$

- This results in a Gaussian wave packet centered around $x = 0$.

Fourier Transform of Gaussian

- The inverse Fourier transform yields:

$$\psi(x) = A_0' e^{-\frac{1}{2}(\Delta k)^2 x^2} e^{ik_0 x}$$

- This is a Gaussian wave packet in x-space with width $\Delta x = \frac{1}{\Delta k}$.

Relating Δx and Δk

- From the Fourier analysis:

$$\Delta x \cdot \Delta k \geq \frac{1}{2}$$

- Since $p = \hbar k$, we substitute:

$$\Delta x \cdot \Delta p = \Delta x \cdot \hbar \Delta k \geq \frac{\hbar}{2}$$

- This is the Heisenberg uncertainty principle.

Implications

- A well-localized particle in space (small Δx) has a large momentum uncertainty Δp.
- A sharply defined momentum results in a broad spatial distribution.
- This reflects the wave-particle duality of quantum objects.

Summary

- The uncertainty principle is not merely a measurement limitation.
- It arises from the fundamental structure of wave mechanics.
- Gaussian wave packets offer the minimum uncertainty product: $\Delta x \Delta p = \frac{\hbar}{2}$.

References

- Griffiths, "Introduction to Quantum Mechanics"
- Shankar, "Principles of Quantum Mechanics"
- French and Taylor, "An Introduction to Quantum Physics"

Problem Set 1

1. An electron is accelerated through a potential difference of 100 V.
   (a) Calculate its de Broglie wavelength.
2. Calculate the de Broglie wavelength of a neutron at room temperature $T = 300\,\mathrm{K}$. Take neutron mass $m_n = 1.675 \times 10^{-27}$ kg, and average thermal energy $E = \frac{3}{2}kT$.

Problem Set 2

3. A cricket ball of mass 0.15 kg moves with a speed of 30 m/s. Calculate its de Broglie wavelength.
   Comment on its observability.
4. Electrons with a de Broglie wavelength of 0.1 nm are used to observe crystal structures.
   (a) Find the accelerating potential required to achieve this wavelength.
   (b) Compare this to typical X-ray wavelengths.

Problem Set 3

- Compare the de Broglie wavelengths of a proton and electron moving with the same kinetic energy of 1 keV.
- Estimate the minimum kinetic energy of an electron confined within a nucleus of radius $r \approx 1.0 \times 10^{-14}$ m.
  Comment on whether an electron can exist in the nucleus.

Problem Set 4

- A particle of mass $m = 1.0 \times 10^{-27}$ kg is confined in a one-dimensional box of length $L = 10^{-10}$ m.
  Use the uncertainty principle to estimate the ground state energy of the particle.
- The velocity of an electron is measured to an accuracy of $1000\,\mathrm{m/s}$.
  Estimate the minimum uncertainty in its position.

Problem Set 5

9. A particle decays with a mean lifetime of 1.0×10^{-8} s.
   Estimate the energy width ΔE of the particle's spectral line using time-energy uncertainty.
10. A wave packet has a spatial width $\Delta x = 0.2\,\mu$m.
    Estimate the minimum momentum uncertainty Δp, and hence calculate the spread in velocity for an electron.