CLASSICAL MECHANICS (M.Sc. Ist YEAR) 1

Review Of Newtonian Mechanics

We know eq. of motion

$$\vec{F} = m\vec{a}$$ (valid for systems with constant mass)

$$= m\ddot{\vec{r}} = m\frac{d^2\vec{r}}{dt^2}$$

$$\vec{F} = m\vec{a} = m\frac{d\vec{v}}{dt}$$

$$\vec{F} = m\frac{d\vec{v}}{dt} = m\frac{d\vec{v}}{dr}\left(\frac{d\vec{r}}{dt}\right)$$ (multiplying & dividing by dr)

$$\vec{F} = m\vec{v}\frac{d\vec{v}}{dr}$$

In 1-dimension motion $\vec{v} = \frac{d\vec{x}}{dt}$, $\vec{r} = \vec{x}$

$$\vec{F} = m\vec{v}\frac{d\vec{v}}{dx}$$

$$\boxed{F = mv\frac{dv}{dx}}$$

Now, work done in moving mass 'm' from position $1 \longrightarrow 2$ is

$$W_{12} = \int_1^2 \vec{F}\, d\vec{r}$$

$$= \int_1^2 m\vec{v}\frac{d\vec{v}}{dr}\, d\vec{r}$$

$$= m\int_1^2 \vec{v}\, d\vec{v}$$

$$= m\left|\frac{v^2}{2}\right|_1^2$$

$$W_{12} = \frac{1}{2}m(v_2^2 - v_1^2) \quad \text{—— (A)}$$

For conservative force $\vec{F}$

$$\vec{\nabla}\times\vec{F} = 0$$

the above relation is true only if

$$\vec{F} = -\vec{\nabla}\phi$$

where ϕ = potential fn

$$\Rightarrow \quad \vec{\nabla}\times\vec{F} = \vec{\nabla}\times\vec{\nabla}\phi = 0$$

So. $$W_{12} = \int_1^2 \vec{F}\, d\vec{r} = -\int_1^2 \vec{\nabla}\phi \cdot d\vec{r}$$ $\left\{ \begin{array}{l} \text{Remember} \\ d\vec{r} = dx\hat{i} + dy\hat{j} + dz\hat{k} \end{array}\right.$

$$= \int_1^2 \left(\frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz\right)$$ (ϕ = is f of x,y,z)

$$= \int_1^2 d\phi = (\phi_2 - \phi_1) \quad \text{— (B)}$$

from eq (A) & (B)

$\Rightarrow$

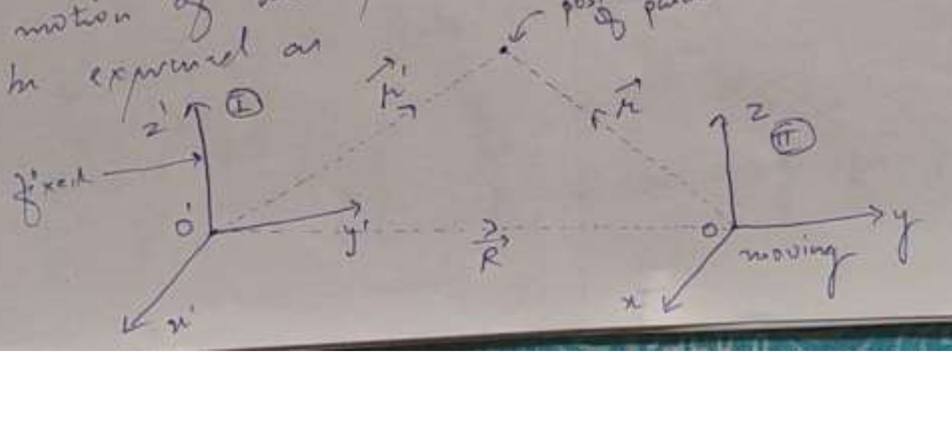
$$\boxed{\frac{1}{2}mv_1^2 + \phi_1 = \frac{1}{2}mv_2^2 + \phi_2}$$

$\Rightarrow$ When a system of constant mass 'm' move in a conservative field (or force) 'F' from position (1) to position (2) then total mechanical energy (K.E + P.E) at position (1) is same as on position (2)

—— x ——

Discussion About Inertial Frame :-

Let us consider a particle moving with force F. The motion of this particle in two inertial frames can be expressed as



Now we can write

$$\vec{r}' = \vec{R} + \vec{r}$$

$$\frac{d\vec{r}'}{dt} = \frac{d\vec{R}}{dt} + \frac{d\vec{r}}{dt}$$

Now as frame (I) is fixed & frame (II) is moving w.r.t (I)

$$\dot{\vec{r}}' = \vec{V} + \dot{\vec{r}}$$

$$\ddot{\vec{r}}' = \dot{\vec{V}} + \ddot{\vec{r}}$$

if V is constant, then $\dot{\vec{V}} = 0$

$$\ddot{\vec{r}}' = \ddot{\vec{r}} \quad (\ddot{r} = \text{acceleration})$$

so

⇒ If frame (II) is moving with constant velocity w.r.t frame (I), then eq. of motion of particle will remain invariant w.r.t frame of reference.

Different types of forces

1) $\vec{F} = \text{constant}$

2) $\vec{F} = \vec{F}(t)$ (like $F = F_0 \cos\omega t$)

3) $\vec{F} = \vec{F}(r)$ ($\vec{F} = \frac{\vec{r}}{r^3}$, central force)

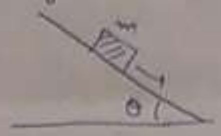
4) $\vec{F} = \vec{F}(\vec{v})$ ($\vec{F} = k\vec{v}$, viscous drag force)

CONSTRAINTS

Anythings restricts the motion of a particle is a constraint

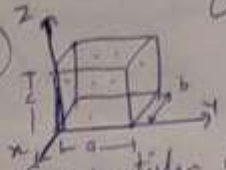
Ex:-

(I)



mass on angular plane
eq. of motion will be
$ax+by+cz=p$

(II)



Gas particles in a box.

$0 \le x \le -b$
$0 \le y \le a$
$0 \le z \le c$
for particle lying on yz plane

(III)

pendulum motion

$x^2+y^2 = l^2$ for rigid rod

$x^2+y^2 \le l^2$ for string

Constraints which can be expressed in equations are called holonomic constraints.
In above examples case (I) & (III) are holonomic constraints.

Case (II) is non-holonomic constraints where constraints are expressed as inequalities & not equations.

Holonomic Constraints

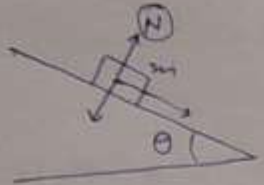
- Rheonomic (depend on time)
- Scleronomic (independent of time) Ex:- Case (III)

$$\vec{F_i} = \dot{\vec{p_i}} \rightarrow \vec{F_i} - \dot{\vec{p_i}} = 0 \rightarrow \sum_i (\vec{F_i} - \dot{\vec{p_i}}) \cdot d\vec{r_i} = 0$$

force — momentum for i^{th} particle

$$\vec{F_i}^{ext} + \vec{F_i}^{int} = \dot{\vec{p_i}}$$

F^{ext} = External force
F^{int} = Internal force



for a man on inclined plane (N) is the normal reaction force which contributes for $\vec{F_i}^{int}$

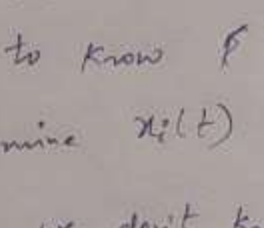
If there is friction present, it will also add to in $\vec{F_i}^{int}$.
However, if we consider no friction force, $\vec{F_i}^{int}$ is only due to (N) which is a constraint.

$$\vec{F_i}^{ext} + \vec{F_i}^{int} = \dot{\vec{p_i}} = m_i \frac{d^2 x_i}{dt^2}$$

To solve problem like above, we need to know F information about F_i^{ext}, F_i^{int}, m_i, to determine $x_i(t)$

But there are problems sometime, where we don't know F_i^{int}. Ex:- Rigid body problem

Rigid body is a collection of particles with that mutual distance between them is fixed.



i.e $(\vec{r_i} - \vec{r_j})^2 = C_i^2$ C = constant

take derivative $r_{ij} = (r_i - r_j)$

$$2\vec{r_{ij}}\, d\vec{r_{ij}} = 0$$

or $F_{ij} \cdot dr_{ij} = 0$ for F to be central force

⇒ Work done in a rigid body is always zero.

$\boxed{\text{Forces of constraint in holonomic constraints systems do not perform any work. (as they are acting } \perp \text{ to motion in most cases)}}$

⇒ For, Holonomic + Scleronomic constraints

$$\vec{r} \perp \vec{F}$$

so $dW = \vec{F} \cdot \vec{dr} = 0$

Also For, Holonomic + Rheonomic Constraints

$$\vec{r} \text{ not } \perp \vec{F}$$

so $\vec{dW} = \vec{F} \cdot \vec{dr} \neq 0$

But we want work done (dW) to be zero for all holonomic systems.

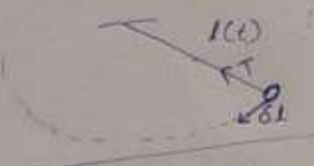
This can be achieved by <u>VIRTUAL DISPLACEMENT (VD)</u>

and by <u>VIRTUAL WORK (VW)</u>

VD is

1) Displacement consistent with constraint forces
2) There is no passage of time

Consider a pendulum of length $l(t)$ (which is fn of time).



At any instant the pendulum is at position as shown.

Since, at this instant time is not considered to be moving

$l(t) = \text{constant}$

This is holonomic + rheonomic system i.e why $l(t)$

And with $l(t) = \text{constant}$, consider there is small displacement (virtual) δl.

Now, since $l(t) = \text{constant}$, the displ. δl will be along the path of circle of fixed radius.

i.e δl will always be $\perp$ to tension T in the string

In rheonomic constraint $l(t)$ is changing with time, so the motion of bob will not be in a circle of fixed radius but spirally reducing circle.

i.e displ is always $\perp$ to force of constraint.

Virtual Work (VW) = Force × displ (virtual)

$\therefore$

$$\delta W = F^c \cdot \delta l$$

$$\delta W = T \cdot \delta l$$

Since $T \perp \delta l$

$$\boxed{\delta W = 0}$$

$\Rightarrow$ Virtual displ. leads to virtual work which is always zero.

With virtual displ, the work done in Rehonomic constraint systems can be zero.

we know $\vec{F} = \dot{\vec{p}}$ $\quad p$ = momentum

for system of particles

$$\sum_i \vec{F_i} = \sum_i \dot{\vec{p_i}}$$

$$\sum_i (\vec{F_i} - \dot{\vec{p_i}}) = 0$$

δr_i = virtual displ.

or $$\sum_i (\vec{F_i} - \dot{\vec{p_i}}) \cdot \delta \vec{r_i} = 0$$

Now $$\vec{F_i} = \vec{F}^a + \vec{F}^c$$

$\vec{F}^a$ = applied force

$\vec{F}^c$ = force of constraints

$$\sum_i (\vec{F}^a + \vec{F}^c - \dot{\vec{p_i}}) \cdot \delta \vec{r_i} = 0$$

but $\vec{F}^c \cdot \delta \vec{r_i} = 0$ (virtual work)

$\Rightarrow$ $$\boxed{\sum_i (\vec{F}^a - \dot{\vec{p_i}}) \cdot \delta \vec{r_i} = 0} \rightarrow \text{D'Alembert's Principle}$$

$\Rightarrow$ we don't have to worry about force of constraints & work done by them. Only work done by applied forces have significance.

$$\boxed{\sum_{i=1}^{3N} (F_i^a - \dot{p}_i) \cdot \delta x_i = 0}$$

$i = 1, 2, 3 \ldots \ldots 3N$

Generalised Coordinates :-

Let us consider a system of N ~~point~~ particles. So no. of co-ordinates required to specify N particles is 3N.

Let the system have K constraints. So the effective no. of co-ordinates will be 3N − K.

Suppose 3N − K = n

Let this n no. of independent co-ordinates can be written as

$q_1, q_2, q_3 \cdots q_n$

Now, these new set of co-ordinates $q_1, q_2 \cdots q_n$ should have relation the (3N−K) co-ordinate.

like
$$r_1 = r_1(q_1\, q_2 \cdots q_n)$$
$$r_2 = r_2(q_1\, q_2 \cdots q_n)$$
$$r_3 = r_3(q_1\, q_2 \cdots q_n)$$
$$\cdots\cdots$$
$$r_{3N} = r_{3N}(q_1, q_2 \cdots q_n)$$

A1

We know D'Alembert's Principle

$$\sum_{i=1}^{3N} (\vec{F}_i^a - \dot{\vec{p}}_i) \cdot \delta x_i = 0$$

where x_i are standard co-ordinates

We would like to reduce (express) D'Alembert's principle in terms of generalized co-ordinates $3N - K = n$ where $n = q_1 q_2 \cdots q_n$ independent variables. So that

$\boxed{K = \text{constraints}}$

$$C_1 x_1 + C_2 x_2 + C_3 x_3 \cdots C_n x_n = 0$$

if and only if $x_1, x_2, \ldots x_n$ are independent

& $C_1, C_2 \cdots C_n = 0$

$\boxed{\text{Here } C_i = (\vec{F}_i - \dot{\vec{p}}_i)}$

Now we know for holonomic constraints, we can express them in terms equations. Hence we can express constraint k as $\phi_\alpha (x_1\ x_2 \cdots x_{3N}, t) = 0$

$\alpha \rightarrow 1 \text{ to } K$

we can write $d\phi_\alpha = 0$

or $$\sum_{i=1}^{3N} \frac{\partial \phi_\alpha}{\partial x_i} dx_i + \frac{\partial \phi_\alpha}{\partial t} dt = 0$$

for virtual displ. δx_i

$$\sum_{i=1}^{3N} \frac{\partial \phi_\alpha}{\partial x_i} \delta x_i + \frac{\partial \phi_\alpha}{\partial t} \delta t = 0$$

$$\delta \phi_\alpha + 0 = 0$$

where $\delta \phi_\alpha = \frac{\partial \phi_\alpha}{\partial x_i} \delta x_i$

& $\frac{\partial \phi_\alpha}{\partial t} \delta t = 0$

$\Rightarrow$

A2

or $\quad \lambda_\alpha \delta \phi_\alpha = 0 \quad$ λ_α = arbitrary constant

Now, since above quantity is a null quantity we can add it to the D'Alembert's principle without loosing ~~any~~ any generality

i.e $$\sum_{i=1}^{3N} \left[(F_i^a - \dot{p}_i) + \lambda_\alpha \delta \phi_\alpha \right] \delta x_i = 0$$

or $$\sum_{i=1}^{3N} \left(F_i^a - \dot{p}_i + \sum_\alpha \lambda_\alpha \frac{\partial \phi_\alpha}{\partial x_i} \right) \delta x_i = 0$$

Now above eq. can be expressed as

$$C_1 \delta x_1 + C_2 \delta x_2 + C_3 \delta x_3 \ldots + C_{3N} \delta x_{3N} = 0$$

since $\delta x_1, \delta x_2 \ldots \delta x_{3N}$ are not independent, we can not say $C_1, C_2 \ldots C_{3N}$ vanishes independently.

Now if we can adjust λ_α in such a way that k no. of values vanishes automatically, then we have reduced $3N$ co-ordinates to $3N-k$ co-ordinates. (which we want to achieve)

where $3N - k = n$ $\quad$ n = independent paramet

Now we have n no. of independent parameters each ~~separat~~ seperatly vanishes.

i.e $$F_i^a - \dot{p}_i + \sum_\alpha \lambda_\alpha \frac{\partial \phi_\alpha}{\partial x_i} = 0$$

The relation is called Lagrange's eq. of I^{st} Kind
& λ_α = Lagrangian Undetermined Multiplier

we know D' Alembert's Principle

$$\sum_{i=1}^{3N} (F_i - \dot{p}_i)\delta x_i = 0$$

Now ~~first~~ second term in above eq. we can write

$$\dot{p}_i \delta x_i = \sum_i m_i \ddot{x}_i \delta x_i \qquad \text{where } \delta x_i = \frac{\partial x_i}{\partial q_k}\delta q_k$$

$$= \sum_{i,k} m_i \ddot{x}_i \frac{\partial x_i}{\partial q_k}\delta q_k$$

$$= \frac{d}{dt}\left(m_i \dot{x}_i \frac{\partial x_i}{\partial q_k}\delta q_k\right) - m_i \dot{x}_i \frac{d}{dt}\left(\frac{\partial x_i}{\partial q_k}\right) \quad (A)$$

Now we can write

$$dx_i = \frac{\partial x_i}{\partial q_k} dq_k + \frac{\partial x_i}{\partial t} dt$$

$$\frac{dx_i}{dt} = \frac{\partial x_i}{\partial q_k}\frac{dq_k}{dt} + \frac{\partial x_i}{\partial t}$$

$$\dot{x}_i = \frac{\partial x_i}{\partial q_k}\dot{q}_k$$

$$\boxed{\frac{\partial \dot{x}_i}{\partial \dot{q}_k} = \frac{\partial x_i}{\partial q_k}}$$

put in eq- (A)

$$= \left[\frac{d}{dt}\left(m_i \dot{x}_i \frac{\partial \dot{x}_i}{\partial \dot{q}_k}\right) - m_i \dot{r}_i \frac{d}{dt}\left(\frac{\partial r_i}{\partial q_k}\right)\right]\delta q_k$$

$$= \left[\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right) - m_i \dot{r}_i \frac{d}{dt}\left(\frac{\partial r_i}{\partial q_k}\right)\right]\delta q_k$$

$$= \left[\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_k} - m_i \dot{r}_i\left(\frac{\partial \dot{r}_i}{\partial q_k}\right)\right]\delta q_k$$

T = Kinetic Energy

$$= \frac{1}{2} m_i \dot{x}_i^2 = \frac{1}{2} m_k \dot{q}_k^2$$

Also,

$$\frac{d}{dt}\frac{\partial r_i}{\partial q_k} = \frac{\partial \dot{r}_i}{\partial q_k}$$

or we can write

$$= \left[\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_k} - m_i \dot{x}_i \frac{\partial \dot{x}_i}{\partial q_k}\right]\delta q_k$$

$$= \left[\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_k} - m_i v_i \frac{\partial v_i}{\partial q_k}\right]\delta q_k$$

$$\dot{p}_i \delta x_i = \left[\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_k} - \frac{\partial T}{\partial q_k}\right]\delta q_k$$

$$\Rightarrow \quad \dot{p}_i \delta x_i = \left[\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_k} - \frac{\partial T}{\partial q_k}\right]\delta q_k \quad \text{——— (B)}$$

So, IInd term in D' Alembert's principle can be expressed as above.

Now I^{st} term in D'Alembert's principle

$$\vec{F_i} \cdot \delta x_i = \vec{F_i} \frac{\partial x_i}{\partial q_k} \delta q_k$$

Also $\vec{F_i} = -\vec{\nabla}_i V$

or let us consider ~~$\frac{F_i \cdot \delta x_i}{\partial}$~~

$$\left[F_i \frac{\partial x_i}{\partial q_k} \right] = Q_k$$

where Q_k = generalised force

$$F_i \cdot \delta x_i = Q_k \delta q_k \quad \text{— (C)}$$

so $Q_k \delta q_k$ = generalised work

Combining eq. (B) & (C)

$$\left[\frac{d}{dt} \frac{\partial T}{\partial \dot{q}_k} - \frac{\partial T}{\partial q_k} - Q_k \right] \delta q_k = \sum_{i=1}^{3N} (F_i - \dot{p}_i) \delta x_i = 0$$

OR

$$\sum_{i=1}^{3N} (F_i - \dot{p}_i) \delta x_i = 0 = \left[\frac{d}{dt} \frac{\partial T}{\partial \dot{q}_k} - \frac{\partial T}{\partial q_k} - Q_k \right] \delta q_k$$

OR

$$\boxed{\left[\frac{d}{dt} \frac{\partial T}{\partial \dot{q}_k} - \frac{\partial T}{\partial q_k} - Q_k \right] \delta q_k = 0} \quad \text{— (D)}$$

D' Alembert's Principle in generalised co-ordinates.

Now, $Q_k = -\dfrac{\partial V}{\partial q_k}$ V= potential in generalised system $V = V(q_1, q_2, \dots q_k)$

eq. D becomes

$$= \left[\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right) - \frac{\partial}{\partial q_k}(T-V)\right]\delta q_k \quad \text{—— (E)}$$

Since V is a potential which is a function of generalised co-ordinates $q_1, q_2, \dots q_k$,
then $\dfrac{\partial V}{\partial \dot{q}_k} = 0$ as potential does not depend on velocity $\dot{q}_k$

so we can write eq. E as

$$= \left[\frac{d}{dt}\frac{\partial (T-V)}{\partial \dot{q}_k} - \frac{\partial}{\partial q_k}(T-V)\right]\delta q_k$$

$$= \left(\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_k} - \frac{\partial}{\partial q_k}L\right)\delta q_k$$ where $L = T-V$
L= Lagrangian

or D' Alembert's Principle in generalised co-ordinates can be written as

$$\sum_k \left(\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_k} - \frac{\partial L}{\partial q_k}\right)\delta q_k = 0$$

Here we can write

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} = 0 \quad \text{—— (F)}$$

which represents $\underset{(3N-k)}{\overset{}{n}}$ no. of independent eq.

$n = (3N-k)$

which are called Euler-Lagrange's eq. of 2nd kind.

Example - I

Consider a case of free particle.

A free particle has 3 degree of freedoms.

so, A K.E of free particle $T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$

P.E " " " $V = 0$

So Lagrangian $L = T - V$

$L = T - 0$

$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$

put in eq. (F) we get

$$\frac{d}{dt}(2\dot{x}) - 0 = 0$$

$$\frac{d}{dt}(2\dot{y}) - 0 = 0$$

$$\frac{d}{dt}(2\dot{z}) - 0 = 0$$

$$\Rightarrow \quad \ddot{x} = 0 \;,\; \ddot{y} = 0 \;,\; \ddot{z} = 0$$

which is true for a free particle.

A free particle has no force acting on it & hence acceleration is zero.

Example - II

Central Orbits

$$T = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) \;,\; V = V(r)$$

$$L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - V(r)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r} = 0$$

$$\frac{d}{dt}(m\dot{r}) - mr\dot{\theta}^2 - F(r) = 0$$

$$m\ddot{r} - mr\dot{\theta}^2 = F(r)$$

$$\boxed{m(\ddot{r} - r\dot{\theta}^2) = F(r)}$$

||ly

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} = 0$$

$$\frac{d}{dt} mr^2\dot{\theta} = 0$$

$$\boxed{mr^2\dot{\theta} = \text{constant}}$$

we can see that θ does not appear in the Lagrangian explicitly. Such generalised co-ordinates are called CYCLIC-CO-ORDINATES.

CENTRAL FORCES

Family of force field with following properties

→ Force field pointing towards or out of a fixed point

→ Magnitude of force along the line towards or away from fixed point

$|\vec{F}| = \vec{F}(\vec{r})$ i.e magnitude of force will be fn of dis r

→ Such forces are conservative by nature. (Energy, E = const)

→ Angular momentum is conserved i.e $\vec{L}$ = const

→ Motion is planar i.e motion of particle under the influence of central force only, then is planar or confined to a plane.

Examples :- 1) Gravitational force

2) Coulomb attraction / repulsion OR Electrostatic force

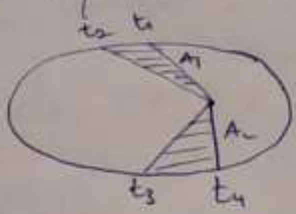
3) Spring force OR force due to a spring

KEPLER'S LAWS

1) Path / orbit of each planet is an ellipse with sun at one foci

2) The line joining the sun with planet, sweeps out equal area in equal interval of time

$$\frac{A_1}{t_2 - t_1} = \frac{A_2}{t_4 - t_3}$$



3) If a planet has time period T, then T is related to semi-major axis of ellipse by relation

$$T^2 \propto a^3$$

Proff of Kepler's Laws

1) In case of inverse sq. force law i.e $F \propto \frac{1}{r^2}$ closed orbit is ellipse. the most general case of i.e first law is correct

2) IInd is law states that $\frac{dA}{dt}$ = const. i.e areal velocity is constant

$$dA = \frac{1}{2} r \cdot r d\theta$$

$$\frac{dA}{dt} = \frac{1}{2} r^2 \frac{d\theta}{dt}$$

$$\frac{dA}{dt} = \frac{1}{2} r^2 \dot{\theta}$$

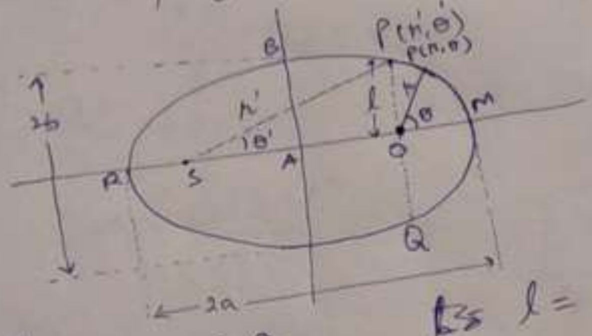
Now $r^2 \dot{\theta}$ for any central orbit is a constant equal to $\frac{L}{m}$

$$\frac{dA}{dt} = \frac{1}{2} \frac{L}{m} = \text{Constant}$$

$$\frac{dA}{dt} = \text{Constant}$$

3) Third law states that

$$T^2 \propto a^3$$ where a = semi-major axis



as l = semi-latus rectum

$PQ = 2l$ = latus rectum

$$\frac{l}{r} = 1 + \epsilon \cos\theta$$

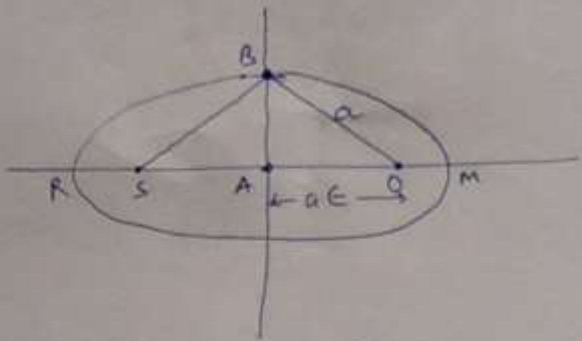
for $\epsilon = 0 \rightarrow$ circle

$0 < \epsilon < 1 \rightarrow$ ellipse

$\epsilon = 1 \rightarrow$ parabola

$\epsilon > 1 \rightarrow$ hyperbola

At M, $r + r' = 2a$



$OB + SB = 2OB = 2a$

$OB = a$

$OA = AM - OM$

$OA = a - OM$

Now $OM = \frac{l}{1+\epsilon}$

and $a = \frac{l}{1-\epsilon^2}$

$OM = \frac{a(1-\epsilon^2)}{1+\epsilon}$

$OM = a(1-\epsilon)$

$\Rightarrow$ $OA = a - a(1-\epsilon)$

$\boxed{OA = a\epsilon}$

so, $AB = \sqrt{a^2 - a^2\epsilon^2}$

$AB = a\sqrt{1-\epsilon^2}$

$b = a\sqrt{1-\epsilon^2}$

$OM = \frac{l}{1+\epsilon}$

IIly $OR = \frac{l}{1-\epsilon}$

$OM + OR = 2a$

$\frac{l}{1+\epsilon} + \frac{l}{1-\epsilon} = 2a$

$2a = \frac{2l}{1-\epsilon^2}$

$\boxed{a = \frac{l}{1-\epsilon^2}}$

Now $T = \frac{\text{area}}{\text{areal velocity}}$

$T = \frac{\pi a b}{L/2m}$

$T = \frac{\pi a^2 \sqrt{1-\epsilon^2}}{L/2m}$

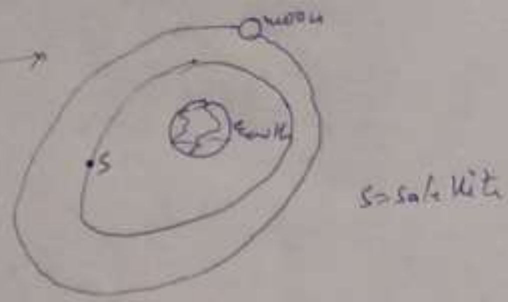
$T^2 = \frac{4m^2\pi^2}{L^2} a^4 (1-\epsilon^2)$

Now $l = a(1-\epsilon^2)$

$$(1-\epsilon^2) = \frac{l}{a}$$

$$T^2 = \left[\frac{4m^2\pi^2}{L^2} l\right] a^3$$

$$\boxed{T^2 \propto a^3}$$



$T_m \sim 28$ days

$$T^2 \propto a^3$$

$$\left(\frac{T_M}{T_s}\right)^2 = \left(\frac{a_m}{a_s}\right)^3$$

If we know three parameters, fourth one can be calculated