Modern Physics:Lecture 06: Finite Potential Well

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Outline

Motivation

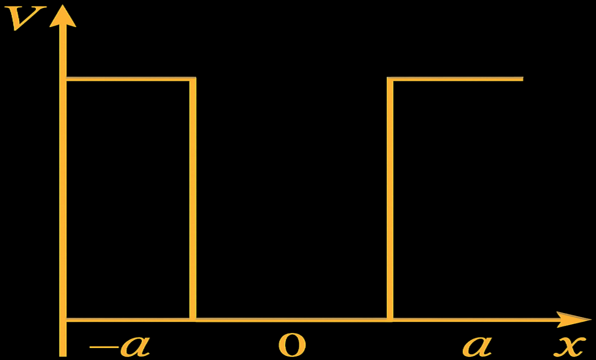
- Fundamental model in quantum mechanics.
- Appears in nanotechnology, quantum dots, nuclear physics, and semiconductor physics.
- Demonstrates quantization, tunneling, and wave-particle duality.

Finite vs Infinite Well

- Infinite square well: walls are impenetrable.
- Finite well: allows wavefunction to penetrate barriers (tunneling).
- More realistic for physical systems.

Potential Profile

$$V(x) = \begin{cases} 0 & |x| > a \\ -V_0 & |x| \leq a \end{cases}$$



Schrödinger Equation

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V(x)\psi = E\psi$$

- Time-independent, one-dimensional case.

Region I $(x < -a)$

$$\frac{d^2\psi}{dx^2} = \kappa^2\psi, \quad \kappa = \frac{\sqrt{-2mE}}{\hbar}$$

$$\psi_I(x) = Ae^{\kappa x}, \quad \text{discard growing term}$$

Region III ($x > a$)

$$\psi_{III}(x) = Fe^{-\kappa x}, \quad \kappa = \frac{\sqrt{-2mE}}{\hbar}$$

- Exponentially decaying wavefunction.

Region II ($|x| \leq a$)

$$\frac{d^2\psi}{dx^2} = -k^2\psi, \quad k = \frac{\sqrt{2m(E + V_0)}}{\hbar}$$

$$\psi_{II}(x) = B\cos(kx) + C\sin(kx)$$

Parity Consideration

- Potential is even: $V(-x) = V(x)$.
- Solutions are either even or odd functions.
- Even: $\psi(x) = B\cos(kx)$
- Odd: $\psi(x) = C\sin(kx)$

Matching Conditions (Even Case)

$$B\cos(ka) = Fe^{-\kappa a}$$
$$-Bk\sin(ka) = -F\kappa e^{-\kappa a}$$

$$\Rightarrow k\tan(ka) = \kappa$$

Matching Conditions (Odd Case)

$$C\sin(ka) = Fe^{-\kappa a}$$
$$Ck\cos(ka) = -F\kappa e^{-\kappa a}$$

$$\Rightarrow -k\cot(ka) = \kappa$$

Dimensionless Parameters

$$\xi = ka, \quad \eta = \kappa a$$

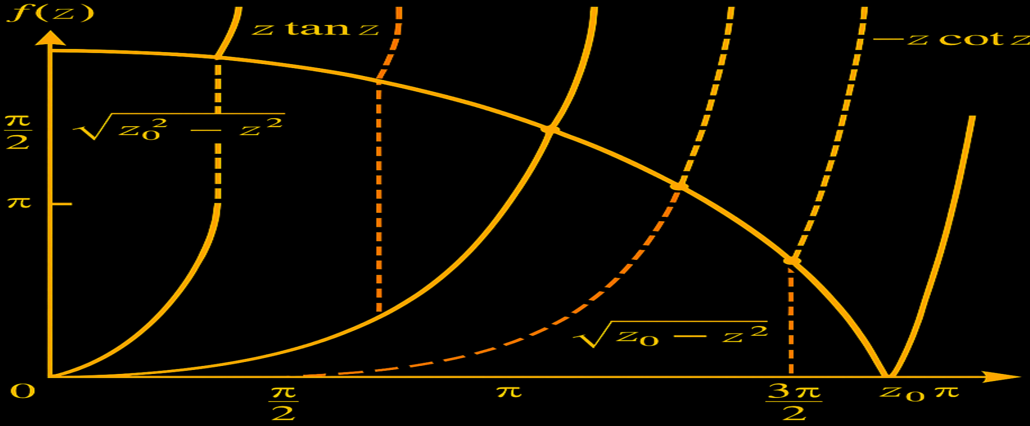
$$\Rightarrow \xi^2 + \eta^2 = \left(\frac{2mV_0a^2}{\hbar^2}\right) = z^2$$

- Useful for graphical solutions.

Transcendental Equations

- Even: $\xi \tan \xi = \eta$
- Odd: $-\xi \cot \xi = \eta$
- Subject to: $\xi^2 + \eta^2 = z^2$

Graphical Solution Method

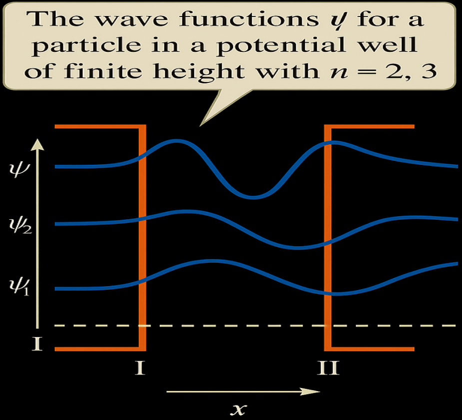


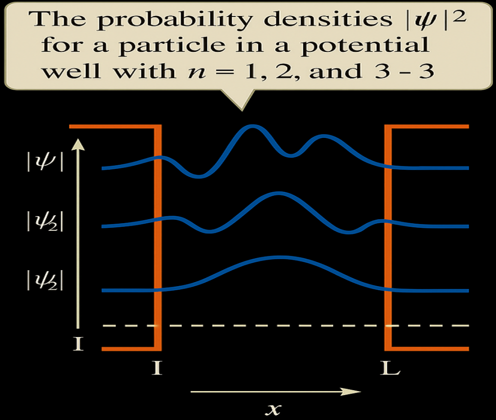
- Plot LHS and RHS of transcendental equations.

Energy Spectrum

- Number of solutions depends on well depth and width.
- Energy levels are discrete.
- No analytical formula for general case.

Wave Function





Wavefunction Behavior

- Oscillatory inside well.
- Exponentially decaying outside.
- Continuous and smooth.

Classical vs Quantum

- Classical: particle confined between $\pm a$.
- Quantum: non-zero probability outside well.
- Quantum tunneling.

Tunneling

- Wavefunction does not vanish outside well.
- Important in nuclear decay, quantum devices.
- Direct consequence of finite potential.

Limiting Case: Infinite Well

- $V_0 \to \infty$: $\kappa \to \infty$
- $\psi = 0$ outside, sinusoidal inside.
- Recovers infinite square well result.

Experimental Relevance

- Quantum wells: semiconductor heterostructures.
- Quantum dots: confined systems in 3D.
- STM: relies on tunneling effect.

Numerical Example

- Parameters: $V_0 = 50$ eV, $a = 0.5$ nm
- Calculate allowed energy levels numerically.
- Plot wavefunctions and potential.

Summary

- Finite well shows key quantum phenomena.
- Solving yields discrete energy levels.
- Appears in modern quantum technologies.

References

- D. J. Griffiths, *Introduction to Quantum Mechanics*
- B. H. Bransden, C. J. Joachain, *Quantum Mechanics*
- R. Eisberg, R. Resnick, *Quantum Physics*

Problem 1: Ground State Energy

Given: $V_0 = 50\,\mathrm{eV}, \quad a = 0.2\,\mathrm{nm}, \quad m = m_e$

- Write the transcendental equation for even bound states:

$$k\tan(ka) = \kappa$$

- Where:

$$k = \sqrt{\frac{2m|E|}{\hbar^2}}, \quad \kappa = \sqrt{\frac{2m(V_0 - |E|)}{\hbar^2}}$$

- Estimate allowed energies graphically or numerically.

Problem 2: Number of Bound States

Determine: The number of bound states

- Use the condition:

$$\frac{\sqrt{2mV_0}}{\hbar}a > \frac{\pi}{2}(n-1)$$

- Count the number of solutions to the transcendental equations:

$$k\tan(ka) = \kappa \quad \text{(even)}$$
$$-k\cot(ka) = \kappa \quad \text{(odd)}$$

Problem 3: Bound State Energies

Task: Solve for the first two energy levels

- Numerically solve the transcendental equations
- Verify that the energies are below V_0
- Compare even and odd states

Problem 4: Probability Density

Find: $|\psi(x)|^2$ for the ground state

- Write general wavefunction in three regions
- Normalize the wavefunction
- Plot probability density and interpret

Problem 5: Effect of Well Width

Vary width a for fixed V_0

- Study how the number of bound states change
- Analyze how energy levels shift
- Discuss implications for quantum confinement

Problem 6: Classical Turning Points

Task: Locate turning points for a given energy E

- For $E < V_0$, turning points at $x = \pm a$
- Compute penetration depth outside well:

$$\delta = \frac{1}{\kappa} = \frac{\hbar}{\sqrt{2m(V_0 - |E|)}}$$

- Interpret physically