Modern Physics:Lecture 04: Quantum Foundations

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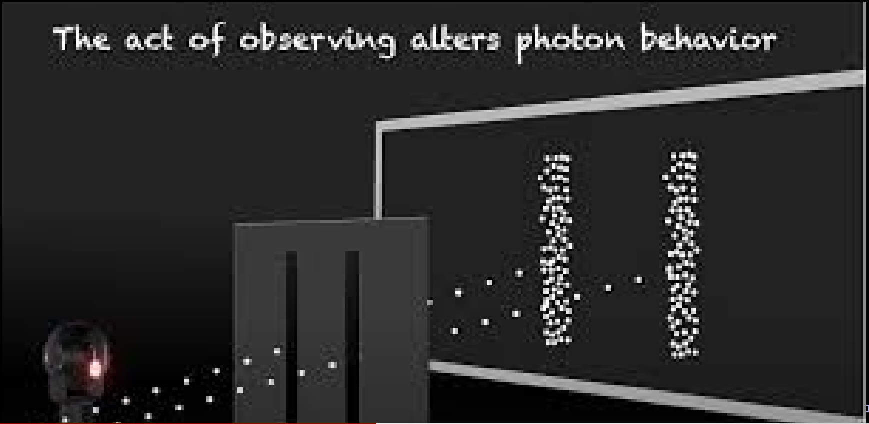
Outline

The Double-Slit Experiment with Electrons

- Consider a source emitting electrons towards a barrier with two slits.
- Beyond the slits, a detection screen records arrival positions.
- Classically, we expect two bands behind the slits.
- Quantum mechanically, we observe an interference pattern.

Quantum Mystery

- Electrons arrive as individual particles.
- Over time, they form an interference pattern.
- Closing one slit removes the interference pattern.
- Measurement of which slit destroys interference.



Postulate 1: The Wavefunction

- The state of a quantum system is fully described by a wavefunction $\psi(x, t)$.
- $\psi(x, t)$ contains all information about the system.

$$\int |\psi(x,t)|^2 \, dx = 1 \quad \text{(Normalization)}$$

Why is the Wavefunction Complex?

- Interference requires phase information.
- Only complex functions carry both magnitude and phase.
- Example: Superposition $\psi = \psi_1 + \psi_2$.
- Probability depends on modulus squared: $|\psi|^2$.

Wavefunction Interpretation

- $|\psi(x,t)|^2$: probability density
- Must be single-valued, continuous, and normalizable.
- Example: Particle in a box

Normalization and Probability

$$\int_{-\infty}^{\infty} |\psi(x,t)|^2 \, dx = 1$$

- Ensures total probability is 1.
- Unnormalized wavefunctions can be scaled.

Postulate 2: Observables as Operators

- Every measurable quantity corresponds to a linear operator.
- Examples:

$$\hat{p} = -i\hbar\frac{d}{dx}, \quad \hat{x} = x$$

Physical Quantities and Their Operators

Quantity	Operator	Notes
Position x	$\hat{x} = x$	Multiplication
Momentum p	$\hat{p} = -i\hbar\frac{d}{dx}$	Position representation
Kinetic Energy T	$\hat{T} = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2}$	
Potential Energy V	$\hat{V} = V(x)$	
Hamiltonian H	$\hat{H} = \hat{T} + \hat{V}$	Total energy
Angular Momentum L_z	$\hat{L}_z = -i\hbar\frac{\partial}{\partial\phi}$	Cylindrical coords
Angular Momentum L^2	$\hat{L}^2 = -\hbar^2\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]$	Spherical coords

Properties of Operators

- Linearity: $\hat{A}(a\psi + b\phi) = a\hat{A}\psi + b\hat{A}\phi$
- Operators must yield real expectation values for physical observables.

Expectation Values

- The expected value of observable A:

$$\langle A \rangle = \int \psi^*(x) \hat{A} \psi(x)\, dx$$

- Example: Momentum

$$\langle p \rangle = \int \psi^* (-i\hbar \frac{d}{dx}) \psi\, dx$$

Postulate 3: Measurement Outcomes

- Only eigenvalues of operators are possible outcomes.
- Measurement affects the wavefunction.

Postulate 4: Time Evolution

- Time evolution governed by the Schrödinger equation:

$$i\hbar \frac{\partial \psi}{\partial t} = \hat{H}\psi$$

- $\hat{H}$ is the Hamiltonian operator.

Time-Dependent Schrödinger Equation

The full time-dependent Schrödinger equation:

$$i\hbar\frac{\partial\psi(x,t)}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\psi(x,t)}{\partial x^2} + V(x)\psi(x,t)$$

- Assume potential is time-independent: $V = V(x)$.
- Use separation of variables: $\psi(x,t) = \phi(x)T(t)$.

Applying Separation of Variables

Substitute $\psi(x,t) = \phi(x)T(t)$ into the Schrödinger equation:

$$i\hbar\phi(x)\frac{dT(t)}{dt} = -\frac{\hbar^2}{2m}T(t)\frac{d^2\phi(x)}{dx^2} + V(x)\phi(x)T(t)$$

Divide both sides by $\phi(x)T(t)$:

$$i\hbar\frac{1}{T(t)}\frac{dT(t)}{dt} = -\frac{\hbar^2}{2m}\frac{1}{\phi(x)}\frac{d^2\phi(x)}{dx^2} + V(x)$$

- Left side depends on t, right side on x.
- Each side must equal a constant, denoted E.

Separated Equations

We now obtain two ordinary differential equations:

Time-Independent Schrödinger Equation

$$-\frac{\hbar^2}{2m}\frac{d^2\phi(x)}{dx^2} + V(x)\phi(x) = E\phi(x)$$

Time Equation

$$i\hbar\frac{dT(t)}{dt} = ET(t)$$

Solution:

$$T(t) = e^{-iEt/\hbar}$$

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Complete Solution

- The total wavefunction is:
$$\psi(x,t) = \phi(x)e^{-iEt/\hbar}$$
- Time dependence appears as a global phase factor.
- Probability density $|\psi(x,t)|^2 = |\phi(x)|^2$: time-independent.
- Used to describe stationary states.

Physical Meaning of Schrödinger's Equation

- Predicts how wavefunctions evolve over time.
- Preserves normalization: total probability conserved.
- Central postulate in non-relativistic quantum theory.

Conclusion

- Quantum mechanics departs from classical determinism.
- Wavefunction and measurement are core ideas.
- Schrödinger equation governs evolution.
- Probabilistic nature reflects fundamental principles.

Wave Function Normalisation

Problem 1:
Normalize the wave function $\psi(x) = Ae^{-\alpha x^2}$, where $\alpha > 0$.

Problem 2:
Determine the normalization constant for $\psi(x) = A \sin\left(\frac{n\pi x}{L}\right)$ for $0 < x < L$.

Problem 3:
Show that the momentum operator $\hat{p} = -i\hbar\frac{d}{dx}$ is Hermitian.

Problem 4:
Find the commutator $[\hat{x}, \hat{p}]$ and verify the canonical commutation relation.

Expectation Values

Problem 5:
Given $\psi(x) = Ae^{-\alpha x^2}$, calculate $\langle x \rangle$.

Problem 6:
Calculate $\langle p \rangle$ for the same Gaussian wave packet $\psi(x) = Ae^{-\alpha x^2}$.

Problem 7:
Write the time-dependent wave function for a stationary state $\psi_n(x, t)$ of a particle in an infinite potential well.

Problem 8:
Show that the total probability remains conserved under time evolution using the Schrödinger equation.

Measurement and Uncertainty

Problem 9:
Calculate the uncertainty Δx and Δp for the Gaussian wave packet $\psi(x) = Ae^{-\alpha x^2}$.

Problem 10:
Verify the Heisenberg uncertainty principle $\Delta x \Delta p \geq \frac{\hbar}{2}$ for this wave packet.

References and Further Reading

- Griffiths, D.J., *Introduction to Quantum Mechanics*
- Beiser, A., *Concepts of Modern Physics*
- Feynman Lectures on Physics

Thank You!

Questions and Discussion